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A MEAN-VALUE THEOREM OF THE RIEMANN ZETA FUNCTION

By F. T. WANG (*Hangchow*)

[Received 30 July 1945]

1. It is well known that, for any fixed $\sigma > \frac{1}{2}$,

$$\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^2\,dt \sim \zeta(2\sigma) \quad (T \to \infty).$$

I shall establish in this note the following analogous theorem:

$$\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^2 \log|\zeta(\sigma+it)|\,dt \sim \zeta(2\sigma)\log\zeta(2\sigma).$$

Proof. By Jensen's inequality,* when we take

$$P = |\zeta|^{2-\epsilon}, \quad f = |\zeta|^{\epsilon}, \quad \phi(x) = x\log x,$$

we get

$$\int_0^T |\zeta|^2 \log|\zeta|\,dt \geqslant \frac{1}{\epsilon}\int_0^T |\zeta|^2\,dt \log\frac{\int_0^T |\zeta|^2\,dt}{\int_0^T |\zeta|^{2-\epsilon}\,dt}. \tag{1}$$

Let us put

$$C_k(\sigma) = \prod_p \left\{\sum_{\nu=0}^{\infty} \binom{\nu+k-1}{\nu}^2 p^{-2\sigma\nu}\right\}. \tag{2}$$

By a theorem due to Titchmarsh and Ingham,† for an arbitrary given positive fixed number ϵ we can determine a number T_ϵ such that

$$\left|\frac{1}{T}\int_0^T |\zeta|^{2-\epsilon}\,dt - C_{1-\frac{1}{2}\epsilon}(2\sigma)\right| < \epsilon^2$$

when $T > T_\epsilon$. Hence

$$\int_0^T |\zeta|^{2-\epsilon}\,dt = TC_{1-\frac{1}{2}\epsilon}(2\sigma) + O(T\epsilon^2) \tag{3}$$

when $T > T_\epsilon$. From (1) and (3) we have

$$\int_0^T |\zeta|^2 \log|\zeta|\,dt \geqslant \frac{1}{\epsilon}\{T\zeta(2\sigma)+O(\epsilon T)\}\log\frac{\zeta(2\sigma)+O(\epsilon^2)}{C_{1-\frac{1}{2}\epsilon}(2\sigma)+O(\epsilon^2)}$$

$$\geqslant T\zeta(2\sigma)\frac{1}{\epsilon}\log\frac{\zeta(2\sigma)}{C_{1-\frac{1}{2}\epsilon}(2\sigma)} + O(T\epsilon) \tag{4}$$

when $T > T_\epsilon$.

* Hardy, Littlewood, and Pólya (1), 151. † Titchmarsh (3), Ingham (2).

Now
$$\binom{\nu-\frac{1}{2}\epsilon}{\nu}^2 = \left(1-\frac{\epsilon}{2}\right)^2\left(1-\frac{\epsilon}{4}\right)^2\ldots\left(1-\frac{\epsilon}{2\nu}\right)^2$$
$$= 1-\epsilon L(\nu)+O(\epsilon^2 2^\nu \log \nu),$$
where
$$L(\nu) = 1+\tfrac{1}{2}+\ldots+1/\nu.$$

Therefore
$$\sum_p \log \frac{\sum\limits_0^\infty p^{-2\sigma\nu}}{\sum\limits_{\nu=0}^\infty \binom{\nu-\frac{1}{2}\epsilon}{\nu}^2 p^{-2\sigma\nu}} = \epsilon \sum_p \frac{\sum\limits_{\nu=1}^\infty L(\nu)p^{-2\sigma\nu}}{\sum\limits_{\nu=0}^\infty p^{-2\sigma\nu}} + O(\epsilon^2) \tag{5}$$
since $\sum\limits_p \sum\limits_{\nu=1}^\infty p^{(1-2\sigma)\nu} \log \nu$ is convergent.

By (2) and (5) we have
$$\log \frac{\zeta(2\sigma)}{C_{1-\frac{1}{2}\epsilon}(2\sigma)} = \sum_p \log \frac{\sum\limits_{\nu=0}^\infty p^{-2\sigma\nu}}{\sum\limits_{\nu=0}^\infty \binom{\nu-\frac{1}{2}\epsilon}{\nu}^2 p^{-2\sigma\nu}}$$
$$= \epsilon \sum_p \left\{\sum_{\nu=1}^\infty \frac{1}{\nu} p^{-2\sigma\nu}\right\} + O(\epsilon^2)$$
$$= \epsilon \log \zeta(2\sigma) + O(\epsilon^2). \tag{6}$$

Hence, when $T > T_\epsilon$, we have
$$\int_0^T |\zeta|^2 \log|\zeta|\, dt \geqslant T\zeta(2\sigma)\log\zeta(2\sigma) + O(T\epsilon).$$

Taking the successive limits $T \to \infty$ and $\epsilon \to 0$, we have
$$\varliminf_{T\to\infty} \frac{1}{T} \int_0^T |\zeta|^2 \log|\zeta|\, dt \geqslant \zeta(2\sigma)\log\zeta(2\sigma). \tag{7}$$

I now use a general inequality concerning the arithmetical and geometrical mean,* in which I take
$$p = |\zeta|^2, \quad f = |\zeta|, \quad r = \epsilon.$$

Then
$$\frac{\int_0^T |\zeta|^2 \log|\zeta|\, dt}{\int_0^T |\zeta|^2\, dt} \leqslant \log \left\{\frac{\int_0^T |\zeta|^{2+\epsilon}\, dt}{\int_0^T |\zeta|^2\, dt}\right\}^{\frac{1}{\epsilon}}.$$

* Hardy, Littlewood, and Pólya (**1**), Th. 184.

Hence $$\int_0^T |\zeta|^2 \log|\zeta|\, dt \leqslant T\zeta(2\sigma)\frac{1}{\epsilon}\log\frac{C_{1+\frac{1}{2}\epsilon}(2\sigma)}{\zeta(2\sigma)}+O(T\epsilon)$$

when $T > T_\epsilon$.

As in (6), we similarly get

$$\int_0^T |\zeta|^2 \log|\zeta|\, dt \leqslant T\zeta(2\sigma)\log\zeta(2\sigma)+O(T\epsilon)$$

when $T > T_\epsilon$. Hence

$$\varlimsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta|^2 \log|\zeta|\, dt \leqslant \zeta(2\sigma)\log\zeta(2\sigma). \tag{8}$$

The theorem follows from (7) and (8).

2. Hardy and Littlewood* proved that, when $\sigma = \frac{1}{2}$,

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\, dt \sim T\log T.$$

I cannot extend my method to establish the analogue. The following inequality, however, is true for sufficiently large values of T:

$$A_1 T\log T\log_2 T \leqslant \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \log|\zeta(\tfrac{1}{2}+it)|\, dt \leqslant A_2 T\log T\log_2 T.$$

In general, we can prove that, when $0 < k \leqslant 2$ and $\sigma > \frac{1}{2}$,

$$\int_0^T |\zeta(\sigma+it)|^{2k}\log|\zeta(\sigma+it)|\, dt \sim TC_{2k}(2\sigma)\sum_p \frac{\sum_{\nu=1}^{\infty}\binom{\nu+k-1}{\nu}E(\nu)p^{-2\nu\sigma}}{\sum_{\nu=0}^{\infty}\binom{\nu+k-1}{\nu}^2 p^{-2\nu\sigma}},$$

where $$E(\nu) = \sum_{\mu=1}^{\nu}\frac{\mu+k-1}{\mu^2}.$$

* Titchmarsh (4).

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2. A. E. Ingham, 'Mean-value theorems and the Riemann zeta function': *Quart. J. of Math.* (Oxford), 4 (1933), 278–90.
3. E. C. Titchmarsh, 'Mean-value theorems in the theory of the Riemann zeta function': *Messenger of Math.* 58 (1929), 125–9.
4. —— *The Zeta Function of Riemann* (Cambridge, 1932).

ON THE ZEROS OF THE RIEMANN ZETA FUNCTION

By E. C. TITCHMARSH (*Oxford*)

[Received 18 June 1946]

1. LET $N(T)$ denote the number of zeros $\beta+i\gamma$ of $\zeta(s)$ such that $0<\gamma\leqslant T$, and let $N_0(T)$ denote the number of zeros such that $\beta=\frac{1}{2}$, $0<\gamma\leqslant T$. The Riemann hypothesis is equivalent to the statement that $N_0(T)=N(T)$ for all values of T.

It is known that $N(T)\sim(2\pi)^{-1}T\log T$ as $T\to\infty$. It was proved by Hardy and Littlewood* that

$$N_0(T)>AT \tag{1.1}$$

(A denotes various absolute constants). Recently, A. Selberg† has shown that

$$N_0(T)>AT\log T. \tag{1.2}$$

This is a remarkable improvement, since it shows that a finite proportion of the zeros of $\zeta(s)$ lie on the line $\mathbf{R}(s)=\frac{1}{2}$. Actually Selberg also proves that

$$N_0(T+U)-N_0(T)>AU\log T \tag{1.3}$$

if $U\geqslant T^a$, where $a>\frac{1}{2}$.

The object of the present paper is to give a shorter proof of (1.2). This is done by using a Fourier transform method similar to that which I used in my Cambridge Tract to prove (1.1). Apart from this, the essential ideas are of course taken from Selberg's paper.

I communicated my proof privately to Dr. Selberg, and in reply he pointed out that a considerable part of it was unnecessary for the proof of (1.2), since it followed the lines of his more detailed analysis used in proving (1.3). The modification due to Dr. Selberg is incorporated in this paper, the analysis suggested by him beginning at § 9.

2. The method of Selberg consists of modifying the series for $\zeta(s)$ by multiplying it by the square of a partial sum of the series for $\{\zeta(s)\}^{-\frac{1}{2}}$. We define α_ν and β_ν by

$$\frac{1}{\sqrt{\zeta(s)}}=\sum_{\nu=1}^{\infty}\frac{\alpha_\nu}{\nu^s}\quad(s=\sigma+it;\ \sigma>1),\qquad \alpha_1=1,$$

* G. H. Hardy and J. E. Littlewood, 'The Zeros of Riemann's Zeta Function on the Critical Line', *Math. Zeitschrift*, 10 (1921), 283–317. See also my Cambridge Tract, *The Zeta Function of Riemann*, Theorem 32.

† A. Selberg, 'On the Zeros of Riemann's Zeta Function', *Norske Videnskaps-Akad. Oslo, Mat.-Naturv. Klasse*, 1942, No. 10, 1–59.

and $$\beta_\nu = \alpha_\nu\left(1-\frac{\log\nu}{\log\xi}\right) \quad (1 \leqslant \nu < \xi).$$

Since $$\frac{1}{\sqrt{\zeta(s)}} = \prod_p \left(1-\frac{1}{p^s}\right)^{\frac{1}{2}} = \prod_p \left\{\sum_{r=0}^{\infty} (-1)^r \binom{\frac{1}{2}}{r} \frac{1}{p^{rs}}\right\},$$

we have $$\alpha_{\nu_1}\alpha_{\nu_2} = \alpha_{\nu_1\nu_2}$$

if $(\nu_1, \nu_2) = 1$. Since the series for $(1-z)^{\frac{1}{2}}$ is majorized by that for $(1-z)^{-\frac{1}{2}}$, we see that, if

$$\sqrt{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\alpha'_\nu}{\nu^s} \quad (\alpha'_1 = 1),$$

then $|\alpha_\nu| \leqslant \alpha'_\nu$ for all values of ν. In particular $|\alpha_\nu| \leqslant 1$, $|\beta_\nu| \leqslant 1$ for all ν.

3. Let $$\phi(s) = \sum_{\nu<\xi} \beta_\nu \nu^{-s},$$

where ξ is to be determined later as a function of δ (or of T). Let

$$\Phi(z) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds \quad (c > 1).$$

There is a pole at $s = 1$ with residue $z\phi(1)\phi(0)$. Hence

$$\Phi(z) = \tfrac{1}{2}z\phi(1)\phi(0) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds.$$

Now, if $s = \frac{1}{2}+it$,

$$\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s) = -2\Xi(t)/(t^2+\tfrac{1}{4}),$$

where $\Xi(t)$ is real for real t. Hence

$$\Phi(z) = \tfrac{1}{2}z\phi(1)\phi(0) - \frac{z^{\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}}|\phi(\tfrac{1}{2}+it)|^2 z^{it}\,dt.$$

On the other hand

$$\Phi(z) = \frac{1}{4\pi i}\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\beta_\mu\beta_\nu \int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\frac{z^s}{n^s\mu^s\nu^{1-s}}\,ds$$

$$= \sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{z^2\nu^2}\right),$$

where μ and ν (and later κ and λ) always run from 1 to ξ. Hence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}|\phi(\tfrac{1}{2}+it)|^2 z^{it}\,dt$$
$$= \tfrac{1}{2}z^{\frac{1}{2}}\phi(1)\phi(0)-z^{-\frac{1}{2}}\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{z^2\nu^2}\right).$$

Putting $z=\exp\{-i(\frac{1}{4}\pi-\frac{1}{2}\delta)-y\}$, it follows that the functions

$$F(t)=\frac{1}{\sqrt{(2\pi)}}\frac{\Xi(t)}{t^2+\frac{1}{4}}|\phi(\tfrac{1}{2}+it)|^2 e^{(\frac{1}{4}\pi-\frac{1}{2}\delta)t},$$

$$f(y)=\tfrac{1}{2}z^{\frac{1}{2}}\phi(1)\phi(0)-z^{-\frac{1}{2}}\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{z^2\nu^2}\right)$$

are Fourier transforms. Since $F(t)$ is real, it follows that

$$\int_{-\infty}^{\infty}\left\{\int_{t}^{t+H}F(u)\,du\right\}^2 dt = 8\int_{0}^{\infty}|f(y)|^2\frac{\sin^2\frac{1}{2}Hy}{y^2}\,dy, \tag{3.1}$$

where H is to be chosen later (it is to be small, and it may be assumed that it is less than 1). The right-hand side is less than

$$2H^2\int_{0}^{1/H}|f(y)|^2\,dy+8\int_{1/H}^{\infty}|f(y)|^2y^{-2}\,dy. \tag{3.2}$$

Putting $y=\log x$, $G=e^{1/H}$, the first integral in (3.2) is equal to

$$\int_{1}^{G}\left|\frac{e^{-i(\frac{1}{4}\pi-\frac{1}{2}\delta)}}{2x}\phi(1)\phi(0)-\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{\nu^2}e^{i(\frac{1}{2}\pi-\delta)}x^2\right)\right|^2dx.$$

Calling the triple sum $g(x)$, this is not greater than

$$2\int_{1}^{G}\frac{|\phi(1)\phi(0)|^2}{4x^2}\,dx+2\int_{1}^{G}|g(x)|^2\,dx<\tfrac{1}{2}|\phi(1)\phi(0)|^2+2\int_{1}^{G}|g(x)|^2\,dx.$$

Similarly the second integral in (3.2) does not exceed

$$\frac{|\phi(1)\phi(0)|^2}{2G\log^2 G}+2\int_{G}^{\infty}\frac{|g(x)|^2}{\log^2 x}\,dx.$$

We have to obtain upper bounds for these integrals as $\delta\to 0$, but it is more convenient to consider directly the integral

$$J(X,\theta)=\int_{X}^{\infty}|g(x)|^2x^{-\theta}\,dx\quad(0<\theta\leqslant\tfrac{1}{2};\ X\geqslant 1).$$

4. We have

$$J(X,\theta)$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\kappa}\sum_{\lambda}\sum_{\mu}\sum_{\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\int\limits_X^\infty \exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2}+\frac{n^2\mu^2}{\nu^2}\right)x^2\sin\delta+\right.$$
$$\left.+i\pi\left(\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right)x^2\cos\delta\right\}\frac{dx}{x^\theta}. \quad (4.1)$$

Let $\sum_1$ denote the sum of those terms in which $m\kappa/\lambda = n\mu/\nu$, and $\sum_2$ the remainder. Let $(\kappa\nu,\lambda\mu) = q$, so that

$$\kappa\nu = aq, \qquad \lambda\mu = bq, \qquad (a,b) = 1.$$

Then, in $\sum_1$, $ma = nb$, so that $n = ra$, $m = rb$ $(r = 1,2,\ldots)$. Hence

$$\sum\nolimits_1 = \sum_{\kappa}\sum_{\lambda}\sum_{\mu}\sum_{\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\sum_{r=1}^{\infty}\int\limits_X^\infty \exp\left(-2\pi\frac{r^2\kappa^2\mu^2}{q^2}x^2\sin\delta\right)\frac{dx}{x^\theta}.$$

Now

$$\sum_{r=1}^{\infty}\int\limits_X^\infty e^{-r^2x^2\eta}\frac{dx}{x^\theta} = \eta^{\frac{1}{2}\theta-\frac{1}{2}}\sum_{r=1}^{\infty}\frac{1}{r^{1-\theta}}\int\limits_{Xr\sqrt{\eta}}^\infty e^{-y^2}\frac{dy}{y^\theta}$$
$$= \eta^{\frac{1}{2}\theta-\frac{1}{2}}\int\limits_{X\sqrt{\eta}}^\infty \frac{e^{-y^2}}{y^\theta}\left(\sum_{r\leqslant y/(X\sqrt{\eta})}\frac{1}{r^{1-\theta}}\right)dy.$$

The last r-sum is of the form

$$\frac{1}{\theta}\left(\frac{y}{X\sqrt{\eta}}\right)^\theta-\frac{1}{\theta}+K(\theta)+O\left\{\left(\frac{y}{X\sqrt{\eta}}\right)^{\theta-1}\right\},$$

where $K(\theta)$, and later $K_1(\theta)$, are bounded functions of θ. Hence we obtain

$$\frac{1}{\theta X^\theta\eta^{\frac{1}{2}}}\left\{\int\limits_0^\infty e^{-y^2}\,dy+O(X\sqrt{\eta})\right\}-\frac{\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta}\left[\int\limits_0^\infty e^{-y^2}y^{-\theta}\,dy+O\{(X\sqrt{\eta})^{1-\theta}\}\right]+$$
$$+\eta^{\frac{1}{2}\theta-\frac{1}{2}}K(\theta)\left[\int\limits_0^\infty e^{-y^2}y^{-\theta}\,dy+O\{(X\sqrt{\eta})^{1-\theta}\}\right]+O\{X^{1-\theta}\log(X\sqrt{\eta})\}$$
$$=\frac{\sqrt{\pi}}{2\theta X^\theta\eta^{\frac{1}{2}}}+\frac{K_1(\theta)\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta}+O\left\{\frac{X^{1-\theta}}{\theta}\log(X\sqrt{\eta})\right\}.$$

Putting $\eta = 2\pi\kappa^2\mu^2q^{-2}\sin\delta$, it follows that

$$\sum\nolimits_1 = \frac{S(0)}{2(2\sin\delta)^{\frac{1}{2}}\theta X^\theta}+\frac{K_1(\theta)}{\theta}(2\pi\sin\delta)^{\frac{1}{2}\theta-\frac{1}{2}}S(\theta)+$$
$$+O\left\{\frac{X^{1-\theta}\log(X\sqrt{\eta})}{\theta}\sum_{\kappa}\sum_{\lambda}\sum_{\mu}\sum_{\nu}\frac{|\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu|}{\lambda\nu}\right\}, \quad (4.2)$$

where $$S(\theta)=\sum_{\kappa}\sum_{\lambda}\sum_{\mu}\sum_{\nu}\left(\frac{q}{\kappa\mu}\right)^{1-\theta}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}. \tag{4.3}$$

Let $$\phi_{-\theta}(\rho)=\rho^{1-\theta}\sum_{m|\rho}\frac{\mu(m)}{m^{1-\theta}}=\rho^{1-\theta}\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta}}\right),$$

where $\mu(m)$ is the function of Möbius. Then

$$q^{1-\theta}=\sum_{\rho|q}\phi_{-\theta}(\rho)=\sum_{\rho|(\kappa\nu,\lambda\mu)}\phi_{-\theta}(\rho)=\sum_{\rho|\kappa\nu,\,\rho|\lambda\mu}\phi_{-\theta}(\rho).$$

Hence

$$S(\theta)=\sum_{\rho\leqslant\xi^2}\phi_{-\theta}(\rho)\sum\sum_{\rho|\kappa\nu,\,\rho|\lambda\mu}\sum\sum\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\kappa^{1-\theta}\lambda\mu^{1-\theta}\nu}=\sum_{\rho\leqslant\xi^2}\phi_{-\theta}(\rho)\left(\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu}\right)^2. \tag{4.4}$$

Let d and d_1 denote positive integers whose prime factors divide ρ. Let $\kappa=d\kappa'$, $\nu=d_1\nu'$, where $(\kappa',\rho)=1$, $(\nu',\rho)=1$. Then

$$\sum\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu}=\sum_{\rho|dd_1}\frac{1}{d^{1-\theta}d_1}\sum_{\kappa'\leqslant\xi/d}\frac{\beta_{d\kappa'}}{\kappa'^{1-\theta}}\sum_{\nu'\leqslant\xi/d_1}\frac{\beta_{d_1\nu'}}{\nu'}. \tag{4.5}$$

Now for $(\kappa',\rho)=1$

$$\beta_{d\kappa'}=\alpha_{d\kappa'}\frac{\log\{\xi/(d\kappa')\}}{\log\xi}=\frac{\alpha_d}{\log\xi}\alpha_{\kappa'}\log\frac{\xi}{d\kappa'}.$$

Hence (4.5) is equal to

$$\frac{1}{\log^2\xi}\sum_{\rho|dd_1}\frac{\alpha_d\alpha_{d_1}}{d^{1-\theta}d_1}\sum_{\kappa'\leqslant\xi/d}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta}}\log\frac{\xi}{d\kappa'}\sum_{\nu'\leqslant\xi/d_1}\frac{\alpha_{\nu'}}{\nu'}\log\frac{\xi}{d_1\nu'}. \tag{4.6}$$

5. We now prove three lemmas.

LEMMA α. *We have*

$$\sum_{\kappa'\leqslant\xi/d}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta}}\log\frac{\xi}{d\kappa'}=O\left\{\left(\frac{\xi}{d}\right)^\theta\log^{\frac{1}{2}}\frac{\xi}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right\} \tag{5.1}$$

uniformly with respect to θ.

We may suppose that $\xi\geqslant 2d$, since otherwise the lemma is trivial. Now

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{x^s}{s^2}\,ds=\begin{cases}0 & (0<x\leqslant 1),\\ \log x & (x>1).\end{cases}$$

Also

$$\sum_{(\kappa',\rho)=1}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta+s}}=\prod_{(p,\rho)=1}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{\frac{1}{2}}=\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{\zeta(1-\theta+s)}}.$$

Hence the left-hand side of (5.1) is equal to

$$\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\frac{1}{s^2}\left(\frac{\xi}{d}\right)^s\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-\frac{1}{2}}\frac{ds}{\sqrt{\zeta(1-\theta+s)}}. \tag{5.2}$$

There are singularities at $s=0$ and $s=\theta$. If $\theta \geqslant \{\log(\xi/d)\}^{-1}$, we can take the line of integration through $s=\theta$, the integral round a small indentation tending to zero. Now

$$\left|\frac{1}{\zeta(1+it)}\right| < A|t|$$

for all t (large or small). Also

$$\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-1} = O\left\{\left|\prod_{p|\rho}\left(1+\frac{1}{p^{1-\theta+s}}\right)\right|\right\} = O\left\{\prod_{p|\rho}\left(1+\frac{1}{p}\right)\right\}.$$

Hence (5.2) is

$$O\left\{\left(\frac{\xi}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\int\limits_{-\infty}^{\infty}\frac{|t|^{\frac{1}{2}}\,dt}{\theta^2+t^2}\right\} = O\left\{\left(\frac{\xi}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\frac{1}{\theta^{\frac{1}{2}}}\right\},$$

and the result stated follows.

If $\theta < \{\log(\xi/d)\}^{-1}$, we take the same contour as before modified by a detour round the right-hand side of the circle $|s| = 2\{\log(\xi/d)\}^{-1}$. On this circle

$$|(\xi/d)^s| \leqslant e^2,$$

$$\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-\frac{1}{2}} = O\left\{\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right\}$$

as before, and $$|\zeta(1-\theta+s)| > A\log(\xi/d)$$

since $\zeta(w)$ has a simple pole at $w=1$. Hence the integral round the circle is

$$O\left\{\log^{-\frac{1}{2}}\frac{\xi}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\int\left|\frac{ds}{s^2}\right|\right\} = O\left\{\log^{\frac{1}{2}}\frac{\xi}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right\}.$$

The integral along the part of the line $\sigma=\theta$ above the circle is

$$O\left\{\left(\frac{\xi}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\int\limits_{A(\log\xi/d)^{-1}}^{\infty}\frac{dt}{t^{\frac{3}{2}}}\right\} = O\left\{\left(\frac{\xi}{d}\right)^\theta\log^{\frac{1}{2}}\frac{\xi}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right\}.$$

The lemma is thus proved in all cases.

LEMMA β. $$\sum_{\rho|dd_1}\frac{|\alpha_d\,\alpha_{d_1}|}{dd_1} = O\left\{\frac{1}{\rho}\prod_{p|\rho}\left(1+\frac{1}{p}\right)\right\}.$$

Defining α'_d as in § 2,

$$\sum_{\rho|dd_1} \frac{|\alpha_d \alpha_{d_1}|}{dd_1} \leqslant \sum_{\rho|dd_1} \frac{\alpha'_d \alpha'_{d_1}}{dd_1} = \sum_{\rho|D} \frac{1}{D}$$

(where D is a number of the same class as d or d_1)

$$= \frac{1}{\rho} \prod_{p|\rho} \left(1-\frac{1}{p}\right)^{-1} = O\left\{\frac{1}{\rho} \prod_{p|\rho} \left(1+\frac{1}{p}\right)\right\},$$

the result stated.

Lemma γ.
$$S(\theta) = O\left(\frac{\xi^{2\theta}}{\log \xi}\right)$$

uniformly with respect to θ. In particular

$$S(0) = O\left(\frac{1}{\log \xi}\right).$$

By (4.5), (4.6), and the above lemmas

$$\sum_{\rho|\kappa\nu}\sum \frac{\beta_\kappa \beta_\nu}{\kappa^{1-\theta}\nu} = O\left\{\frac{1}{\log^2\xi} \sum_{\rho|dd_1} \frac{|\alpha_d \alpha_{d_1}|}{d^{1-\theta}d_1} \left(\frac{\xi}{d}\right)^\theta \log^{\frac{1}{2}}\frac{\xi}{d} \log^{\frac{1}{2}}\frac{\xi}{d_1} \prod_{p|\rho} \left(1+\frac{1}{p}\right)\right\}$$

$$= O\left\{\frac{\xi^\theta}{\log \xi} \prod_{p|\rho} \left(1+\frac{1}{p}\right) \sum_{\rho|dd_1} \frac{|\alpha_d \alpha_{d_1}|}{dd_1}\right\}$$

$$= O\left\{\frac{\xi^\theta}{\rho \log \xi} \prod_{p|\rho} \left(1+\frac{1}{p}\right)^2\right\}.$$

Hence

$$S(\theta) = O\left\{\frac{\xi^{2\theta}}{\log^2\xi} \sum_{\rho \leqslant \xi^2} \frac{\phi_{-\theta}(\rho)}{\rho^2} \prod_{p|\rho} \left(1+\frac{1}{p}\right)^4\right\}$$

$$= O\left\{\frac{\xi^{2\theta}}{\log^2\xi} \sum_{\rho \leqslant \xi^2} \frac{1}{\rho^{1+\theta}} \prod_{p|\rho} \left(1+\frac{1}{p}\right)^4\right\}$$

$$= O\left\{\frac{\xi^{2\theta}}{\log^2\xi} \sum_{\rho \leqslant \xi^2} \frac{1}{\rho^{1+\theta}} \sum_{n|\rho} \frac{1}{n^{\frac{1}{2}}}\right\},$$

since

$$\prod_{p|\rho} \left(1+\frac{1}{p}\right)^4 = O\left\{\prod_{p|\rho} \left(1+\frac{4}{p}\right)\right\} = O\left\{\prod_{p|\rho} \left(1+\frac{1}{p^{\frac{1}{2}}}\right)\right\} = O\left(\sum_{n|\rho} \frac{1}{n^{\frac{1}{2}}}\right).$$

Hence

$$S(\theta) = O\left\{\frac{\xi^{2\theta}}{\log^2\xi} \sum_{n \leqslant \xi^2} \sum_{\rho_1 \leqslant \xi^2/n} \frac{1}{(n\rho_1)^{1+\theta} n^{\frac{1}{2}}}\right\} = O\left\{\frac{\xi^{2\theta}}{\log^2\xi} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}+\theta}} \sum_{\rho_1 \leqslant \xi^2/n} \frac{1}{\rho_1^{1+\theta}}\right\}$$

$$= O\left\{\frac{\xi^{2\theta}}{\log^2\xi} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \sum_{\rho_1 \leqslant \xi^2} \frac{1}{\rho_1}\right\} = O\left(\frac{\xi^{2\theta}}{\log \xi}\right).$$

6. By Lemma γ, the first term on the right-hand side of (4.2) is

$$O\left(\frac{1}{\delta^{\frac12}\theta X^{\theta}\log\xi}\right). \tag{6.1}$$

The second term is

$$O\left(\frac{\delta^{\frac12\theta-\frac12}}{\theta}\frac{\xi^{2\theta}}{\log\xi}\right) = O\left(\frac{1}{\delta^{\frac12}\theta X^{\theta}\log\xi}\right)$$

if $X\xi^2 = O(\delta^{-\frac12})$. The third term is

$$O\left\{\frac{X^{1-\theta}\log(X\xi/\delta)}{\theta}\sum_{\kappa}\sum_{\lambda}\sum_{\mu}\sum_{\nu}\frac{1}{\lambda\nu}\right\} = O\left\{\frac{X^{1-\theta}\log(X\xi/\delta)}{\theta}\xi^2\log^2\xi\right\}$$
$$= O\left(\frac{1}{\delta^{\frac12}\theta X^{\theta}\log\xi}\right)$$

provided that
$$X\xi^2 = O\left(\frac{1}{\delta^{\frac12}\log(1/\delta)\log^3\xi}\right). \tag{6.2}$$

We shall ultimately choose ξ so that (6.2) is satisfied. It will then follow from (4.2) that

$$\sum\nolimits_1 = O\left(\frac{1}{\delta^{\frac12}\theta X^{\theta}\log\xi}\right). \tag{6.3}$$

7. Now consider $\sum_2$. If P and Q are positive, and $X \geqslant 1$,

$$\int_{X}^{\infty} e^{-Px^2+iQx^2}\frac{dx}{x^{\theta}} = \frac{1}{2}\int_{X^2}^{\infty}\frac{e^{-Py}}{y^{\frac12\theta+\frac12}}e^{iQy}\,dy = O\left(\frac{e^{-P}}{X^{\theta}Q}\right),$$

e.g. by applying the second mean-value theorem to the real and imaginary parts. Hence

$$\sum\nolimits_2 = O\Bigg[\frac{1}{X^{\theta}}\sum_{\kappa}\sum_{\lambda}\sum_{\mu}\sum_{\nu}\frac{1}{\lambda\nu}\sum_{m}\sum_{n}{}'\left|\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right|^{-1}\times$$
$$\times\exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2}+\frac{n^2\mu^2}{\nu^2}\right)\sin\delta\right\}\Bigg].$$

The terms with $m\kappa/\lambda > n\mu/\nu$ contribute to the (m,n) sum

$$O\left\{\sum_{m=1}^{\infty}e^{-\pi m^2\kappa^2\lambda^{-2}\sin\delta}\sum_{n<m\kappa\nu/\lambda\mu}\left(\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right)^{-1}\right\}.$$

Now
$$\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2} \geqslant \frac{m\kappa}{\lambda}\left(\frac{m\kappa}{\lambda}-\frac{n\mu}{\nu}\right) = \frac{m\kappa(m\kappa\nu-n\lambda\mu)}{\lambda^2\nu},$$

and $$\sum_n \frac{1}{m\kappa\nu-n\lambda\mu} \leqslant 1+\frac{1}{\lambda\mu}+\frac{1}{2\lambda\mu}+\ldots = 1+O\left(\frac{\log m\xi}{\lambda\mu}\right).$$

Hence the (m,n) sum is

$$O\left\{\frac{\lambda^2\nu}{\kappa}\sum_{m=1}^{\infty}\left(\frac{1}{m}+\frac{\log m\xi}{m\lambda\mu}\right)e^{-\pi m^2\kappa^2\lambda^{-2}\sin\delta}\right\}$$
$$= O\left[\frac{\lambda^2\nu}{\kappa}\left\{\left(1+\frac{\log\xi}{\lambda\mu}\right)\log\frac{\lambda^2}{\kappa^2\sin\delta}+\frac{1}{\lambda\mu}\log^2\frac{\lambda^2}{\kappa^2\sin\delta}\right\}\right]$$
$$= O\left\{\frac{\lambda^2\nu}{\kappa}\left(\log\frac{1}{\delta}+\frac{1}{\lambda\mu}\log^2\frac{1}{\delta}\right)\right\}$$

provided that $\log\xi = O(\log 1/\delta)$. The terms with $m\kappa/\lambda < n\mu/\nu$ may be treated similarly. It follows that

$$\sum\nolimits_2 = O\left\{\frac{1}{X^\theta}\sum_\kappa\sum_\lambda\sum_\mu\sum_\nu\left(\frac{\lambda}{\kappa}\log\frac{1}{\delta}+\frac{1}{\kappa\mu}\log^2\frac{1}{\delta}\right)\right\}$$
$$= O\left(X^{-\theta}\xi^4\log\xi\log\frac{1}{\delta}+X^{-\theta}\xi^2\log^2\xi\log^2\frac{1}{\delta}\right)$$
$$= O\left(X^{-\theta}\xi^4\log^2\frac{1}{\delta}\right). \tag{7.1}$$

8. LEMMA δ. *If $\xi = \delta^{-c}$, where c is a constant less than $\frac{1}{8}$, and* (6.2) *is satisfied with $X = G$, then*

$$\int_{-\infty}^{\infty}\left|\int_t^{t+H}F(u)\,du\right|^2 dt = O\left(\frac{H}{\delta^{\frac{1}{2}}\log\xi}\right). \tag{8.1}$$

From (6.3) and (7.1) it follows that

$$J(X,\theta) = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta X^\theta\log\xi}\right) \tag{8.2}$$

uniformly with respect to θ. Hence

$$\int^G |g(x)|^2\,dx = -\int_1^G x^\theta\frac{\partial J}{\partial x}\,dx = [-x^\theta J]_1^G+\theta\int_1^G x^{\theta-1}J\,dx$$
$$= O\left(\frac{1}{\delta^{\frac{1}{2}}\theta\log\xi}\right)+O\left(\theta\int_1^G\frac{dx}{\delta^{\frac{1}{2}}\theta x\log\xi}\right) = O\left(\frac{\log G}{\delta^{\frac{1}{2}}\log\xi}\right),$$

taking e.g. $\theta = \frac{1}{2}$. Also

$$\int_0^{\frac{1}{2}} \theta J(G,\theta)\,d\theta = \int_G^\infty |g(x)|^2\,dx \int_0^{\frac{1}{2}} \theta x^{-\theta}\,d\theta$$

$$= \int_G^\infty |g(x)|^2 \left(\frac{1}{\log^2 x} - \frac{1}{2x^{\frac{1}{2}}\log x} - \frac{1}{x^{\frac{1}{2}}\log^2 x}\right) dx$$

$$\geqslant \int_G^\infty \frac{|g(x)|^2}{\log^2 x}\,dx - \int_G^\infty \frac{|g(x)|^2}{x^{\frac{1}{2}}}\,dx$$

if G is sufficiently large. Hence

$$\int_G^\infty \frac{|g(x)|^2}{\log^2 x}\,dx \leqslant \int_0^{\frac{1}{2}} \theta J(G,\theta)\,d\theta + J(G,\tfrac{1}{2})$$

$$= O\left(\int_0^{\frac{1}{2}} \frac{d\theta}{\delta^{\frac{1}{2}} G^\theta \log \xi}\right) + O\left(\frac{1}{\delta^{\frac{1}{2}} G^{\frac{1}{2}} \log \xi}\right)$$

$$= O\left(\frac{1}{\delta^{\frac{1}{2}} \log G \log \xi}\right).$$

Also $\phi(0) = O(\xi)$, $\phi(1) = O(\log \xi)$. The result therefore follows from the formulae of § 3.

9. So far the integrals considered have involved $F(t)$. We now turn to integrals involving $|F(t)|$. The results about such integrals are expressed in the following lemmas.

LEMMA ϵ. $$\int_{-\infty}^{\infty} |F(t)|^2\,dt = O\left(\frac{1}{\delta^{\frac{1}{2}}} \frac{\log 1/\delta}{\log \xi}\right).$$

By the Fourier-transform formulae, the left-hand side is equal to

$$2\int_0^\infty |f(y)|^2\,dy = 2\int_1^\infty \left|\frac{e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta)}}{2x}\phi(1)\phi(0) - g(x)\right|^2 dx$$

$$\leqslant 4\int_1^\infty |g(x)|^2\,dx + O(\xi^2 \log^2 \xi)$$

in the notation of § 3.

The integral here is $J(1, 0)$. This can be estimated in much the same way as $J(X,\theta)$. We follow the analysis of § 4 until we come to the sum

$$\sum_{r \leqslant y/\sqrt{\eta}} \frac{1}{r} = \log \frac{y}{\sqrt{\eta}} + \gamma + O\left(\frac{\sqrt{\eta}}{y}\right).$$

This leads to a sum similar to $S(0)$, but containing a factor $\log(k\mu/q)$. In dealing with the $\log q$, we have to replace the previous $\phi_{-\theta}(\rho)$ by

$$\left[-\frac{\partial}{\partial\theta}\phi_{-\theta}(\rho)\right]_{\theta=0} = \left(\log\rho+\sum_{p|\rho}\frac{\log p}{p-1}\right)\phi(\rho).$$

The factor in brackets is $O(\log\rho) = O(\log 1/\delta)$. Terms of the same order arise from $\log K$ and $\log\mu$. The result is therefore similar to (8.2), but with $1/(\theta X^\theta)$ replaced by $\log 1/\delta$.

This proves the lemma.

LEMMA ζ. $$\int_{-\infty}^{\infty}\left\{\int_t^{t+H}|F(u)|\,du\right\}^2 dt = O\left(\frac{H^2\log\delta^{-1}}{\delta^{\frac{1}{2}}\log\xi}\right).$$

For the left-hand side does not exceed

$$\int_{-\infty}^{\infty}\left\{H\int_t^{t+H}|F(u)|^2\,du\right\}dt = H\int_{-\infty}^{\infty}|F(u)|^2\,du\int_{u-H}^{u}dt = H^2\int_{-\infty}^{\infty}|F(u)|^2\,du,$$

and the result follows from Lemma ϵ.

LEMMA η. *If* $\delta = 1/T$,

$$\int_0^T |F(t)|\,dt > AT^{\frac{3}{4}}.$$

We have $$\left(\int_{\frac{1}{2}+i}^{2+i}+\int_{2+i}^{2+iT}+\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i}\right)\zeta(s)\phi^2(s)\,ds = 0.$$

Now $\phi(s) = O(\xi^{\frac{1}{2}})$ for $\sigma \geqslant \frac{1}{2}$; hence the first term is $O(\xi)$, and the third is $O(\xi T^{\frac{1}{4}})$. Also

$$\zeta(s)\phi^2(s) = 1+\sum_{n=2}^{\infty}\frac{a_n}{n^s},$$

where $|a_n| \leqslant d_3(n)$, the number of ways in which n can be expressed as the product of three factors. Hence

$$\int_{2+i}^{2+iT}\zeta(s)\phi^2(s)\,ds = i(T-1)+\sum_{n=2}^{\infty}a_n\int_{2+i}^{2+iT}\frac{ds}{n^s}$$

$$= i(T-1)+O\left(\sum_{n=2}^{\infty}\frac{d_3(n)}{n^2\log n}\right)$$

$$= iT+O(1).$$

It follows that $$\int_0^T \zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)\,dt = T+o(T).$$

Hence

$$\int_0^T |F(t)|\,dt > A\int_0^T t^{-\frac{1}{4}}|\zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)|\,dt$$

$$> AT^{-\frac{1}{4}}\int_{\frac{1}{2}T}^{T} |\zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)|\,dt$$

$$> AT^{-\frac{1}{4}}\left|\int_{\frac{1}{2}T}^{T} \zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)\,dt\right|$$

$$> AT^{\frac{3}{4}}.$$

Lemma θ.
$$\int_0^T dt\int_t^{t+H} |F(u)|\,du > AHT^{\frac{3}{4}}.$$

The left-hand side is equal to

$$\int_0^{T+H} |F(u)|\,du \int_{\max(0,u-H)}^{\min(T,u)} dt \geqslant \int_H^T |F(u)|\,du\int_{u-H}^{u} dt = H\int_H^T |F(u)|\,du,$$

and the result follows from Lemma η.

10. We can now proceed to the proof of (1.2). Let E be the sub-set of $(0, T)$ where

$$\int_t^{t+H} |F(u)|\,du > \left|\int_t^{t+H} F(u)\,du\right|.$$

If this holds, then $F(u)$ must change sign in $(t, t+H)$; hence so must $\Xi(u)$, and so $\zeta(\frac{1}{2}+iu)$ must have a zero in this interval. Now the above inequality gives

$$\int_E dt\int_t^{t+H} |F(u)|\,du \geqslant \int_E \left\{\int_t^{t+H} |F(u)|\,du - \left|\int_t^{t+H} F(u)\,du\right|\right\} dt$$

$$= \int_0^T \left\{\int_t^{t+H} |F(u)|\,du - \left|\int_t^{t+H} F(u)\,du\right|\right\} dt$$

$$= \int_0^T dt\int_t^{t+H} |F(u)|\,du - \int_0^T \left|\int_t^{t+H} F(u)\,du\right| dt.$$

The left-hand side is not greater than

$$\left\{\int_E dt\int_E\left(\int_t^{t+H} |F(u)|\,du\right)^2 dt\right\}^{\frac{1}{2}} \leqslant \sqrt{m(E)}\left\{\int_{-\infty}^{\infty}\left(\int_t^{t+H} |F(u)|\,du\right)^2 dt\right\}^{\frac{1}{2}}$$

$$< A\{m(E)\}^{\frac{1}{2}}HT^{\frac{1}{4}}$$

by Lemma ζ, with $\delta = 1/T$, $\xi = T^c$. The second term on the right is not greater than

$$\left\{\int_0^T dt \int_0^T \left|\int_t^{t+H} F(u)\,du\right|^2 dt\right\}^{\frac{1}{2}} \leqslant T^{\frac{1}{2}}\left\{\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt\right\}^{\frac{1}{2}}$$

$$< \frac{AT^{\frac{1}{2}}H^{\frac{1}{2}}}{\delta^{\frac{1}{4}}\log^{\frac{1}{2}}\xi} = \frac{AT^{\frac{3}{4}}H^{\frac{1}{2}}}{\log^{\frac{1}{2}}\xi}.$$

Hence
$$\sqrt{m(E)} > A_1\,T^{\frac{1}{2}} - \frac{A_2\,T^{\frac{1}{2}}}{H^{\frac{1}{2}}\log^{\frac{1}{2}}\xi},$$

where A_1 and A_2 denote the particular constants which occur. Taking

$$H = \frac{4A_2^2}{A_1^2\log\xi} = \frac{4A_2^2}{A_1^2\,c\log T}$$

it follows that
$$m(E) > \tfrac{1}{4}A_1^2 T.$$

This choice of H implies that $G = \xi^{\frac{1}{4}A_1^2/A_2^2}$. To satisfy (6.2) for $X \leqslant G$, it is only necessary to take c sufficiently small, but the choice may now depend on A_1 and A_2.

Hence, of the intervals $(0, H)$, $(H, 2H)$,... contained in $(0, T)$, at least $[\frac{1}{4}A_1^2 T/H]$ must contain points of E. Now, if $(nH, (n+1)H)$ contains a point t of E, there must be a zero of $\zeta(\frac{1}{2}+iu)$ in $(t, t+H)$, and so in $(nH, (n+2)H)$. Allowing for the fact that each zero might be counted twice in this way, there must be at least

$$\frac{1}{2}\left[\frac{A_1^2\,T}{4H}\right] \geqslant \frac{A_1^4\,c}{32A_2^2}\,T\log T - \frac{1}{2}$$

zeros in $(0, T)$. This proves the theorem.

TWO LATTICE-POINT PROBLEMS

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1. CONSIDER the class of ellipses $E(f,\Delta)$ which have centre O, area Δ, and, as a pair of conjugate diameters, the given straight lines

$$f(x,y) \equiv ax^2+2hxy+by^2 = 0,$$

where a, h, b are real and the discriminant

$$4h^2-4ab = D^2 \quad (D > 0).$$

Without loss of generality we may take $D = 1$ or any convenient value not zero. For appropriate values of Δ there will be ellipses of the class with the property that on the boundary of any one of them lies a lattice point $P(x,y)$ (and its image) and inside lies no lattice point save O, the origin.

We seek to determine the lower bound $\Delta_0 \geqslant 0$ of such admissible values of Δ. The number Δ_0 is plainly the same for all forms unimodularly equivalent to cf where c is any constant other than zero. We seek also to determine the upper bounds of Δ_0 for all pairs of conjugate directions. For convenience I suppose that neither direction is rational. The analysis is easily modified to include the case of a rational direction.

This problem, as it turns out, is exactly equivalent to the problem of determining the minimum (in the usual sense of lower bound) of the indefinite binary quadratic form f, a problem solved by Markoff, for an exposition of which reference may be made to Dickson (**1**), or the minimum of the product of two homogeneous linear forms.

In three or more dimensions it is equivalent to the problem of finding the minimum of the product of three or more homogeneous linear forms. In each case, of course, the variables take all integral values, the set $(0, 0,..., 0)$ excluded. Plainly, by Minkowski's famous theorem, $\Delta_0 \leqslant 4$.

2. For appropriate Δ there may be an ellipse of the class with the property that on its boundary are two lattice points P, Q not collinear with O (and, of course, their images P' and Q') while inside is no lattice point except O. The problem is to find the lower bound Δ_1 of admissible numbers Δ (naturally Δ_1 is the same for all pairs

unimodularly equivalent to cf) and to find the upper bound of Δ_1 for all pairs of directions.

This problem, it turns out, is equivalent to the following minimum problem for indefinite binary quadratic forms:

Let $$AX^2+2HXY+BY^2 = F \sim cf.$$
To determine $$\min |(A+B)/2H| = \mathfrak{M}(C_f)$$
as F runs through all forms equivalent to cf.

I prove that $$\mathfrak{M} \leqslant \tfrac{1}{2}$$
and that, if equality occurs, then either
$$cf \sim x^2+2xy-2y^2 = f_0$$
or $$cf \sim x^2+4xy-3y^2 = f_1.$$

It then appears that
$$\Delta_1 = \pi/(1-\mathfrak{M}^2)^{\frac{1}{2}}$$
so that $$\pi \leqslant \Delta_1 \leqslant 2\pi/\sqrt{3},$$
both bounds being attained. Whether Δ_1 can take *any* value between these bounds I do not know. It is plausible that there should be a number $d < 2\pi/\sqrt{3}$ such that, if cf is not equivalent to f_0 or f_1, then $\Delta_1 \leqslant d$. But this I cannot prove.

It is worthy of remark that, in this problem on indefinite binary forms, the form x^2+xy-y^2 has relinquished its usual dominant position. Naturally there are corresponding problems in three or more dimensions. But they are very difficult indeed.

3. The first problem. It is a simple question in elementary mathematics to show that the minimum ellipse, with centre O, through a given point $P(x,y)$ with OK, OL as conjugate diameters (P not on OK, OL) has area equal to 2π times the area of the parallelogram formed by OK, OL and the lines through P parallel to OK, OL. Hence, if the equation to OK, OL is
$$f(x,y) \equiv ax^2+2hxy+by^2 = 0, \tag{1}$$
the minimum area is $$\mathfrak{A} = 2\pi|f(x,y)|/D. \tag{2}$$

If then we consider only visible lattice points $P(x,y)$ such that
$$|f(x,y)| < \tfrac{1}{2}D \tag{3}$$
(and infinitely many such lattice points exist), this minimum ellipse has an area less than π. This ellipse cannot have on its boundary any lattice point save P and its image P', nor any lattice point inside

save O. For, if the contrary were the case, it would contain a parallelogram of area $\geqslant 2$ and hence its own area $\geqslant \pi$.

Hence the admissible values of $\Delta < \pi$ are those values of

$$2\pi|f(x,y)|/D < \pi$$

for co-prime integers x, y. Thus

$$\Delta_0 = \min \Delta = 2\pi \min |f(x,y)|/D.$$

Now Markoff showed that

$$\overline{\lim} \min |f(x,y)| = \tfrac{1}{3}$$

and determined all the classes of forms, the *Markoff forms*, for which

$$\min |f(x,y)| > \tfrac{1}{3}.$$

We have therefore $\quad \Delta_0 = \min \Delta \leqslant 2\pi/\sqrt{5},$

equality occurring when

$$cf \sim x^2+xy-y^2.$$

If this class of pairs of directions is excluded, then

$$\Delta_0 \leqslant \pi/\sqrt{2},$$

equality occurring when

$$cf \sim x^2+2xy-y^2,$$

and so on; the precise upper limit of Δ_0 being

$$\overline{\lim}\, \Delta_0 = \tfrac{2}{3}\pi.$$

A complete statement of Markoff's theorems may be found in Dickson (**1**).

4. The second problem.

I now prove

THEOREM 1. *Suppose that*

$$cf \sim AX^2+2HXY+BY^2$$

with $\quad \lambda = |(A+B)/2H| \leqslant \tfrac{1}{2}.$

Then an admissible value of Δ is

$$\Delta = \pi/(1-\lambda^2)^{\frac{1}{2}}.$$

Conversely any admissible value of Δ is of this form.

Suppose that an ellipse, with area Δ and centre O, and having $f = 0$ as conjugate diameters, passes through the two lattice points $P(p_1, p_2)$, $Q(q_1, q_2)$ and has no lattice point inside except O. Plainly P and Q must be visible from O. Also no lattice point can be inside the triangle POQ or between P and Q. Hence

$$\Delta POQ = \tfrac{1}{2}, \qquad p_1 q_2 - p_2 q_1 = \pm 1.$$

Thus, by a unimodular transformation, P and Q become $(1, 0)$ and $(0, 1)$, and the ellipse consequently becomes

$$X^2+2\beta XY+Y^2 = 1, \qquad |\beta| < 1, \tag{1}$$

$$cf \sim AX^2+2HXY+BY^2, \tag{2}$$

$$\Delta = \pi/(1-\beta^2)^{\frac{1}{2}}. \tag{3}$$

The harmonic condition gives

$$A+B = 2H\beta. \tag{4}$$

Since no lattice point other than O lies inside the ellipse, it follows that

$$|2\beta| \leqslant 1, \tag{5}$$

this being necessary and sufficient. Both parts of the theorem follow from (1)–(5).

It may be noted that, if $|2\beta| = 1$, the ellipse will have three lattice points and their images on its boundary.

THEOREM 2. *We have*

$$\Delta_1 = \pi/(1-\mathfrak{M}^2)^{\frac{1}{2}},$$

where $$\mathfrak{M} = \min\lambda = \min |(A+B)/2H|.$$

This follows immediately from Theorem 1, once we have proved

THEOREM 3. *We have* $\qquad \mathfrak{M} \leqslant \frac{1}{2},$

equality occurring when

$$cf \sim x^2+2xy-2y^2 = f_0$$

or $$cf \sim x^2+4xy-3y^2 = f_1,$$

and in no other case.

Corresponding to an indefinite binary quadratic form f without rational factors there is a sequence of positive integers

$$(g_i) \quad (i = ..., -1, 0, 1, 2,...).$$

Write $$F_i = [g_i, g_{i+1},...], \qquad H_i = [g_{i-1}, g_{i-2},...],$$

these being simple continued fractions,

$$A_i = F_i/(F_i H_i+1), \qquad B_i = (F_i H_i-1)/(F_i H_i+1),$$
$$A_{i+1} = H_i/(F_i H_i+1),$$

so that $$B_i^2+4A_i A_{i+1} = 1, \qquad B_i+B_{i+1} = 2g_i A_{i+1},$$
$$\Phi_i(x, y) \equiv (-1)^i A_i x^2+B_i xy+(-1)^{i+1}A_{i+1}y^2.$$

The form f/D is equivalent to any member of this chain of equivalent reduced forms. For an exposition see Dickson (**2**).

Consider now any member of this chain, for example

$$\Phi_0 = A_0x^2+B_0xy-A_1y^2,$$

and with Φ_0 the two forms

$$\phi' = A_0X^2+(2A_0+B_0)XY+(A_0+B_0-A_1)Y^2,$$
$$\phi'' = (A_0-B_0-A_1)X^2+(B_0+2A_1)XY-A_1Y^2$$

obtained by the respective parallel transformations

$$x = X+Y, \quad y = Y; \qquad x = X, \quad y = -X+Y.$$

For these three forms the corresponding values of $|A+B|/|2H|$ are

$$\lambda_0 = |A_0-A_1|/B_0, \qquad \mu = (2A_0+B_0-A_1)/(2A_0+B_0),$$
$$\nu = (2A_1+B_0-A_0)/(2A_1+B_0)$$

since

$$2A_0+B_0-A_1 = [(F_0-1)(H_0+2)+1]/(F_0H_0+1) > 0$$

because $F_0 > 1$, $H_0 > 1$, and similarly $2A_1+B_0-A_0 > 0$.

Suppose that $\qquad \lambda_0 \geqslant \frac{1}{2}, \qquad A_1 \geqslant A_0$

so that

$$2A_1 \geqslant 2A_0+B_0, \qquad \mu = 1-A_1/(2A_0+B_0) \leqslant \tfrac{1}{2},$$

and it may be noted that $\nu > \frac{1}{2}$.

Likewise, if $\lambda_0 \geqslant \frac{1}{2}$, $A_1 \leqslant A_0$, then

$$\nu \leqslant \tfrac{1}{2}, \qquad \mu > \tfrac{1}{2}.$$

Hence one of the three numbers λ_0, μ, ν is less than $\frac{1}{2}$ except possibly when

$$\lambda_0 = \tfrac{1}{2}, \qquad 2|A_1-A_0| = B_0.$$

Thus we have proved the existence of forms of the type required in Theorem 1, and also that $\mathfrak{M} \leqslant \frac{1}{2}$.

5. I now show that, *if the equation*

$$\lambda_i \equiv |A_{i+1}-A_i|/B_i = \tfrac{1}{2} \tag{1, i}$$

holds for five consecutive values of i, *it holds for all* i, *and the only possible forms for which this can happen are* f_0, f_1 *of Theorem* 3.

It will follow that, equivalence to f_0 or f_1 being excluded, there will be an infinity of admissible ellipses with $\Delta < 2\pi/\sqrt{3}$. There is no loss of generality in supposing that $(1, i)$ holds for

$$i = 0, 1, 2, 3, 4. \tag{2}$$

From $(1,i)$ by squaring and using

$$B_i^2+4A_iA_{i+1} = 1 \tag{3}$$

we get $$A_i^2-A_iA_{i+1}+A_{i+1}^2 = \tfrac{1}{4}. \tag{4, i}$$

Take $(4,i)-(4,i+1)$ $(i = 0,\dots,4)$. Then either

$$A_i = A_{i+2} \tag{5, i}$$

or $$A_{i+1} = A_i+A_{i+2}. \tag{6, i}$$

If A_{i+1} lies *between* A_i and A_{i+2}, then $(6,i)$ cannot hold: therefore $(5,i)$ holds. From $(1,i)$ and $(1, i+1)$ and $(5,i)$

$$0 = 2|A_{i+2}-A_i| = 2|A_{i+2}-A_{i+1}|+2|A_{i+1}-A_i|$$
$$= B_{i+1}+B_i,$$

but $$B_i+B_{i+1} = 2g_iA_{i+1}, \tag{7, i}$$

which is positive. Hence A_{i+1} *exceeds both its neighbours or is less than both its neighbours*. Also, *if* $A_{i+1} < A_i$, $A_{i+1} < A_{i+2}$, *then necessarily* $(5,i)$ *holds*, that is $A_i = A_{i+2}$.

(I) Suppose first that $A_1 > A_0$, $A_1 > A_2$. From $(1,0)$, $(1,1)$, $(7,0)$ we obtain

$$4A_1-2A_0-2A_2 = 2g_0A_1, \qquad (2g_0-4)A_1+2A_0+2A_2 = 0$$

so that

$$g_0 = 1, \qquad A_1 = A_0+A_2, \qquad B_0 = 2A_2, \qquad B_1 = 2A_0. \tag{8}$$

Now $A_2 < A_1$ so that also $A_2 < A_3$ and therefore

$$A_1 = A_3, \qquad B_1 = B_2 = g_1A_2. \tag{9}$$

Next $A_3 > A_2$ and so $A_3 > A_4$. Therefore, as above,

$$g_2 = 1, \qquad A_3 = A_2+A_4, \tag{10}$$

so that, by (8), (9), $$A_4 = A_0, \qquad B_3 = B_0. \tag{11}$$

Now $A_4 < A_3$ and therefore $A_4 < A_5$. Hence

$$A_5 = A_3 = A_1, \qquad B_4 = B_0 = B_3 = g_3A_4. \tag{12}$$

Since now Φ_4 is *identical* with Φ_0, the *chain has the period* 4.

Also $$g_3A_0 = B_3 = B_0 = 2A_2,$$
$$g_1A_2 = B_1 = 2A_0,$$

whence $g_1g_3 = 4$, so that

$$g_1 = g_3 = 2 \quad\text{or}\quad g_1 = 1,\ \ g_3 = 4 \quad\text{or}\quad g_1 = 4,\ \ g_3 = 1,$$

giving essentially just two sequences

$$(\overset{\times}{1}, \overset{\times}{2}) \quad \text{and} \quad (\overset{\times}{1}, 1, 1, \overset{\times}{4})$$

and the periods of equivalent reduced forms

$$x^2+2xy-2y^2, \qquad -2x^2+2xy+y^2 \tag{13}$$

$$[1, 4, -3], \quad [-3, 2, 2], \quad [2, 2, -3], \quad [-3, 4, 1] \tag{14}$$

respectively.

(II) The case $A_1 < A_0$, $A_1 < A_2$ can be discussed likewise. It leads to the sequences

$$(\overset{\times}{2}, \overset{\times}{1}) \quad \text{and} \quad (\overset{\times}{4}, 1, 1, \overset{\times}{1}),$$

the new forms being the negatives essentially of (13) and (14).

It remains to show that, for the forms in (13), (14),

$$\min |(A+B)/2H| = \tfrac{1}{2}.$$

Suppose that

$$2|A+B| \geqslant |2H|, \qquad 4H^2-4AB = D^2, \quad D > 0. \tag{15}$$

Then (15) is equivalent to

$$A^2+AB+B^2 \geqslant \tfrac{1}{4}D^2. \tag{16}$$

Now (16) holds automatically if $AB > 0$ and one of $|A|$, $|B|$ is at least $\frac{1}{2}D$. It also holds if $AB < 0$, provided that both $|A|$, $|B|$ are at least $\frac{1}{2}D$. Thus it remains to discuss (16) when one of $|A|$, $|B|$ is at most $\frac{1}{2}D$. Since such a form is equivalent by a parallel transformation to a reduced form, we need only consider a reduced form

$$AX^2+2HXY+BY^2, \qquad |A| \leqslant \tfrac{1}{2}D,$$

and those derived from it by the parallel transformation

$$X = X_1+kY_1, \qquad Y = Y_1,$$

where k is any integer.

For (13) we then get

$$A = 1, \qquad 2H = 2k+2, \qquad B = k^2+2k-2,$$

$$A+B = (k+1)^2-2,$$

but plainly, for any integer m,

$$|(m^2-2)/2m| \geqslant \tfrac{1}{2}.$$

As for (14), there are the two forms

$$x^2+4xy-3y^2, \qquad 2x^2+2xy-3y^2$$

to be considered. For the first of these forms

$$A = 1, \qquad 2H = 2k+4, \qquad B = k^2+4k-3,$$
$$A+B = (k+2)^2-6,$$
$$|(A+B)/2H| = |(m^2-6)/2m| \geqslant \tfrac{1}{2} \quad (m = k+2),$$

and for the second form

$$A = 2, \qquad 2H = 4k+2, \qquad B = 2k^2+2k-3,$$
$$A+B = \tfrac{1}{2}(2k+1)^2-\tfrac{3}{2},$$
$$|(A+B)/2H| = |(m^2-3)/4m| \geqslant \tfrac{1}{2} \quad (m = 2k+1).$$

We have therefore proved the statement at the beginning of the section and have also shown that

$$\mathfrak{M}(f_0) = \tfrac{1}{2}, \qquad \mathfrak{M}(f_1) = \tfrac{1}{2}.$$

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THE APPROXIMATE SOLUTION OF EQUATIONS IN INFINITELY MANY UNKNOWNS

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1. Introduction

In many of the methods used for the approximate solution of integral equations and differential equations the original problem is reduced to that of finding the solution of a system of linear equations in infinitely many unknowns. This system is treated by the method of segments (*réduites*), i.e. only the first m equations are considered, and all save the first m unknowns are neglected in these equations. It is this step which makes the results approximate.

Very little is known about the accuracy of such approximations. This paper tries to remedy partially this lack by some work on the method of segments as applied to eigen-value problems. It is based on the standard (Hilbert) theory of quadratic forms in infinitely many variables (**1, 2, 3**).

Systems of equations of the form

$$x_\alpha - \lambda \sum_{\beta=1}^{\infty} k_{\alpha\beta} x_\beta = 0 \quad (\alpha = 1, 2, \ldots) \tag{1}$$

will be considered. It will be assumed that the matrix $[k_{\alpha\beta}]$ corresponds to a completely continuous bilinear form, i.e. one which satisfies equation (5). In this case equations (1) have properties entirely analogous to those of an algebraic system of m equations in m unknowns.* In general the only solution of (1) is the null solution in which $x_\alpha = 0$ for each α; but for a certain enumerable set of values of λ, the eigen-values, there are non-trivial solutions, the x_α then constituting eigen-vectors.

The concern of this paper is to investigate how closely the eigen-values $\lambda_1^{(m)}, \ldots, \lambda_m^{(m)}$ for the curtailed equations

$$x_\alpha - \lambda \sum_{\beta=1}^{m} k_{\alpha\beta} x_\beta = 0 \quad (\alpha = 1, \ldots, m) \tag{2}$$

approximate to the eigen-values $\lambda_1, \ldots, \lambda_m$ of equations (1). To simplify the discussion it will be assumed that the matrix $[k_{\alpha\beta}]$ is Hermitian ($k_{\alpha\beta} = \bar{k}_{\beta\alpha}$), the eigen-values λ_γ then being real.†

* (2) 94. † (4) 100.

The method adopted in this paper is to solve the equations (1) with $\alpha = m+1,\ m+2,...$ for $x_{m+1},\ x_{m+2},...$ in terms of $x_1,..., x_m$. When $x_{m+1}, x_{m+2},...$ are eliminated from equations (1) with $\alpha = 1,..., m$ one obtains

$$x_\alpha - \sum_{\beta=1}^{m} r_{\alpha\beta}\, x_\beta - \lambda \sum_{\beta=1}^{m} k_{\alpha\beta}\, x_\beta = 0 \quad (\alpha = 1,..., m), \tag{3}$$

where the $r_{\alpha\beta}$ are rather complicated functions of the $k_{\alpha\beta}$ and λ. Obviously equations (3) can only have non-trivial solutions for the eigen-values λ_γ of equations (1). By choosing m sufficiently large, convergent expressions for the $r_{\alpha\beta}$ are obtained for any given λ (because of the complete continuity of $[k_{\alpha\beta}]$). Bounds are readily obtained for the $r_{\alpha\beta}$, and the maximum difference possible between the eigen-values of the exact system (3) and those of the approximate curtailed system (2) can be estimated. To show the convergence of these differences to zero, with increasing m, a slightly stronger assumption than complete continuity seems necessary. In addition it is assumed that

$$\sum_{\alpha=1}^{m} \sum_{\beta=m+1}^{\infty} |k_{\alpha\beta}|^2$$

tends to zero with increasing m. This condition is satisfied, for example, if

$$\sum_{\alpha,\beta=1}^{\infty} |k_{\alpha\beta}|^2$$

converges. Since

$$\sum_{\gamma=1}^{\infty} \frac{1}{(\lambda_\gamma)^2} = \sum_{\alpha,\beta=1}^{\infty} |k_{\alpha\beta}|^2,$$

where the two sides converge or diverge together, the convergence of $\sum_{\alpha,\beta=1}^{\infty} |k_{\alpha\beta}|^2$ is a slightly stronger restriction than that of complete continuity, which implies merely that $1/\lambda_\gamma$ tends to zero with increasing γ.*

Further slight extensions of the results to systems of the form

$$x_\alpha + \sum_{\beta=1}^{\infty} h_{\alpha\beta}\, x_\beta - \lambda \sum_{\beta=1}^{\infty} k_{\alpha\beta}\, x_\beta = 0 \quad (\alpha = 1, 2,...) \tag{4}$$

will be indicated and an example treated.

Throughout it will be assumed that $(x_1, x_2,...)$ is a point in Hilbert space, i.e. that $\sum_{\alpha=1}^{\infty} |x_\alpha|^2$ converges.

* (3) 1558.

2. Preliminary results

This section will give the most important theorems needed later. Proofs are to be found in (**1, 2, 3**). Throughout, the parameter λ will be assumed to be real.

The definition of a *completely continuous* bilinear form $\sum k_{\alpha\beta} x_\alpha x_\beta$ is that it satisfies

$$\left|\sum_{\alpha,\beta=1}^{\infty} k_{\alpha\beta} x_\alpha y_\beta - \sum_{\alpha,\beta=1}^{m} k_{\alpha\beta} x_\alpha y_\beta\right| < \epsilon, \tag{5}$$

for arbitrary positive ϵ, for m sufficiently large, and uniformly for all x_α, y_β satisfying

$$\sum_{\alpha=1}^{\infty} |x_\alpha|^2 = \sum_{\beta=1}^{\infty} |y_\beta|^2 = 1.$$

Then
$$\left|\sum_{\alpha,\beta=m+1}^{\infty} k_{\alpha\beta} x_\alpha y_\beta\right| < \epsilon,$$

where
$$\sum_{\alpha=m+1}^{\infty} |x_\alpha|^2 = \sum_{\beta=m+1}^{\infty} |y_\beta|^2 = 1$$

since the left-hand side is a possible value of the left-hand side of (5). The maximum value of

$$\left|\sum_{\alpha,\beta=m+1}^{\infty} k_{\alpha\beta} x_\alpha y_\beta\right|$$

under the conditions

$$\sum_{\alpha=m+1}^{\infty} |x_\alpha|^2 = 1, \qquad \sum_{\beta=m+1}^{\infty} |y_\beta|^2 = 1$$

will be known as b_m. Therefore, for completely continuous forms,

$$b_m < \epsilon \tag{6}$$

for arbitrary ϵ if m is sufficiently great.

It is well known* that equations (1) can be reduced to the form

$$y_\gamma - (\lambda/\lambda_\gamma) y_\gamma = 0 \quad (\gamma = 1, 2, \ldots) \tag{7}$$

(where the λ_γ occur in order of increasing magnitude) by a unitary transformation

$$y_\gamma = \sum_{\alpha=1}^{\infty} \bar{w}_{\alpha\gamma} x_\alpha. \tag{8}$$

The defining property of a unitary transformation is

$$\sum_{\gamma=1}^{\infty} w_{\alpha\gamma} \bar{w}_{\beta\gamma} = \delta_{\alpha\beta}, \text{ where } \delta_{\alpha\beta} = \begin{cases} 0 & (\alpha \neq \beta), \\ 1 & (\alpha = \beta), \end{cases} \tag{9}$$

* (3) 1562.

which implies
$$\sum_{\gamma=1}^{\infty} w_{\gamma\alpha}\bar{w}_{\gamma\beta} = \delta_{\alpha\beta}. \tag{10}$$

By using (9) the transformation (8) can be inverted, giving
$$x_\alpha = \sum_{\gamma=1}^{\infty} w_{\alpha\gamma} y_\gamma. \tag{11}$$

It is obvious from (7) and (11) that
$$x_{\alpha\gamma} = w_{\alpha\gamma} y_\gamma \tag{12}$$

is the eigen-vector corresponding to the eigen-value λ_γ. Comparison of (1) and (7) gives
$$k_{\alpha\beta} = \sum_{\gamma=1}^{\infty} \frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{\lambda_\gamma}, \tag{13}$$

which is equivalent to
$$\frac{\delta_{\gamma\epsilon}}{\lambda_\gamma} = \sum_{\alpha,\beta=1}^{\infty} \bar{w}_{\alpha\gamma} w_{\beta\epsilon} k_{\alpha\beta}. \tag{14}$$

Some results on the solution of non-homogeneous systems of equations by iteration will now be derived. They are, of course, well known. Consider the system
$$x_\alpha - \lambda \sum_{\beta=1}^{\infty} k_{\alpha\beta} x_\beta = c_\alpha \quad (\alpha = 1, 2, \ldots). \tag{15}$$

By the unitary transformation (8) this is reduced to
$$y_\gamma(1-\lambda/\lambda_\gamma) = c'_\gamma \quad (\gamma = 1, 2, \ldots), \tag{16}$$

where
$$c'_\gamma = \sum_{\alpha=1}^{\infty} \bar{w}_{\alpha\gamma} c_\alpha.$$

Equations (16) have the solution
$$y_\gamma = \frac{c'_\gamma}{1-\lambda/\lambda_\gamma},$$

and thus equations (15) have the solution
$$x_\alpha = \sum_{\beta=1}^{\infty} c_\beta \sum_{\gamma=1}^{\infty} \frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{1-\lambda/\lambda_\gamma}. \tag{17}$$

It is not possible to express
$$\sum_{\gamma=1}^{\infty} \frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{1-\lambda/\lambda_\gamma}$$

directly in terms of the $k_{\alpha\beta}$; but, if $|\lambda/\lambda_\gamma| < 1$ for each γ,

$$\sum_{\gamma=1}^{\infty} \frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{1-\lambda/\lambda_\gamma} = \sum_{\gamma=1}^{\infty}\left\{w_{\alpha\gamma}\bar{w}_{\beta\gamma} + \sum_{n=1}^{\infty}\lambda^n \frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{(\lambda_\gamma)^n}\right\}.$$

The term with $n = 1$ is

$$\lambda\sum_{\gamma=1}^{\infty}\frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{\lambda_\gamma} = \lambda k_{\alpha\beta},$$

by (13). For general n induction is used; we have

$$\begin{aligned}\sum_{\gamma=1}^{\infty}\frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{(\lambda_\gamma)^n} &= \sum_{\gamma,\epsilon=1}^{\infty}\frac{w_{\alpha\gamma}\delta_{\gamma\epsilon}\bar{w}_{\beta\epsilon}}{\lambda_\gamma(\lambda_\epsilon)^{n-1}} \\ &= \sum_{\gamma,\epsilon,\eta=1}^{\infty}\frac{w_{\alpha\gamma}\bar{w}_{\eta\gamma}w_{\eta\epsilon}\bar{w}_{\beta\epsilon}}{\lambda_\gamma(\lambda_\epsilon)^{n-1}} \\ &= \sum_{\epsilon,\eta=1}^{\infty}k_{\alpha\eta}\frac{w_{\eta\epsilon}\bar{w}_{\beta\epsilon}}{(\lambda_\epsilon)^{n-1}}.\end{aligned}$$

Continuing this process gives

$$\begin{aligned}\sum_{\gamma=1}^{\infty}\frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{(\lambda_\gamma)^n} &= \sum_{\eta_1,\ldots,\eta_{n-1}=1}^{\infty}k_{\alpha\eta_1}k_{\eta_1\eta_2}\ldots k_{\eta_{n-1}\beta} \\ &= k_{\alpha\beta}^{(n)}, \quad \text{say.}\end{aligned} \tag{18}$$

It will be seen that

$$\sum_{\gamma=1}^{\infty}\frac{w_{\alpha\gamma}\bar{w}_{\beta\gamma}}{1-\lambda/\lambda_\gamma} = \delta_{\alpha\beta} + \sum_{n=1}^{\infty}\lambda^n k_{\alpha\beta}^{(n)}. \tag{19}$$

Thus (17) can be expressed in the form

$$x_\alpha = \sum_{\beta=1}^{\infty}c_\beta\left\{\delta_{\alpha\beta} + \sum_{n=1}^{\infty}\lambda^n k_{\alpha\beta}^{(n)}\right\} \tag{20}$$

provided that $|\lambda/\lambda_\gamma| < 1$ for each γ. Since the $|\lambda_\gamma|$ are non-decreasing functions of γ, this is equivalent to $|\lambda/\lambda_1| < 1$. The method used in deriving (20) requires the assumption that $[k_{\alpha\beta}]$ is Hermitian. However, the result is true for arbitrary $k_{\alpha\beta}$ provided that $|\lambda|b < 1$, where b is the (Hilbert) bound of $[k_{\alpha\beta}]$.* Most of the theory in this paper follows from (20) and could be extended to $k_{\alpha\beta}$ with no symmetry properties whatsoever.

A few further remarks will be made. The first is that the (Hilbert)

* (2) 92.

bound of the matrix $[k_{\alpha\beta}]$ is $|1/\lambda_1|$. This follows since the bound b is the maximum value of

$$\left|\sum_{\alpha,\beta=1}^{\infty} k_{\alpha\beta}\,\overline{X}_{\alpha}^{(1)} X_{\beta}^{(2)}\right|,$$

where the $X_{\alpha}^{(1)}$, $X_{\alpha}^{(2)}$ are arbitrary apart from the conditions

$$\sum_{\alpha=1}^{\infty} |X_{\alpha}^{(1)}|^2 = \sum_{\beta=1}^{\infty} |X_{\beta}^{(2)}|^2 = 1.$$

But
$$\sum_{\alpha,\beta=1}^{\infty} k_{\alpha\beta}\overline{X}_{\alpha}^{(1)} X_{\beta}^{(2)} = \sum_{\gamma=1}^{\infty} \frac{\overline{Y}_{\gamma}^{(1)} Y_{\gamma}^{(2)}}{\lambda_{\gamma}},$$

where the Y's are related to the X's by the transformation (8). It is easily seen that

$$\sum_{\alpha=1}^{\infty} |X_{\alpha}^{(1)}|^2 = \sum_{\gamma=1}^{\infty} |Y_{\gamma}^{(1)}|^2, \qquad \sum_{\beta=1}^{\infty} |X_{\beta}^{(2)}|^2 = \sum_{\gamma=1}^{\infty} |Y_{\gamma}^{(2)}|^2$$

from the properties of unitary transformations, and it is then obvious that the maximum value of

$$\left|\sum_{\gamma=1}^{\infty} \frac{\overline{Y}_{\gamma}^{(1)} Y_{\gamma}^{(2)}}{\lambda_{\gamma}}\right|$$

is attained when

$$|Y_1^{(1)}| = 1, \qquad |Y^{(2)}| = 1,$$
$$Y_{\gamma}^{(1)} = 0, \qquad Y_{\gamma}^{(2)} = 0 \quad (\gamma > 1)$$

and is equal to $|1/\lambda_1|$. Thus

$$b = \max\left|\sum_{\alpha,\beta=1}^{\infty} k_{\alpha\beta}\,\overline{X}_{\alpha}^{(1)} X_{\beta}^{(2)}\right| = |1/\lambda_1|. \tag{21}$$

By a similar argument upper and lower bounds are obtained for the value of the Hermitian form

$$\sum_{\alpha,\beta=1}^{\infty} \left\{\delta_{\alpha\beta} + \sum_{n=1}^{\infty} \lambda^n k_{\alpha\beta}^{(n)}\right\} \overline{X}_{\alpha} X_{\beta},$$

namely

$$\frac{1}{1+|\lambda/\lambda_1|} \sum_{\alpha=1}^{\infty} |X_{\alpha}|^2 \leqslant \sum_{\alpha,\beta=1}^{\infty} \left\{\delta_{\alpha\beta} + \sum_{n=1}^{\infty} \lambda^n k_{\alpha\beta}^{(n)}\right\} \overline{X}_{\alpha} X_{\beta} \leqslant \frac{1}{1-|\lambda/\lambda_1|} \sum_{\alpha=1}^{\infty} |X_{\alpha}|^2,$$

i.e.

$$\frac{1}{1+|\lambda|b} \sum_{\alpha=1}^{\infty} |X_{\alpha}|^2 \leqslant \sum_{\alpha,\beta=1}^{\infty} \left\{\delta_{\alpha\beta} + \sum_{n=1}^{\infty} \lambda^n k_{\alpha\beta}^{(n)}\right\} \overline{X}_{\alpha} X_{\beta} \leqslant \frac{1}{1-|\lambda|b} \sum_{\alpha=1}^{\infty} |X_{\alpha}|^2. \tag{22}$$

This result is obvious since

$$\sum_{\alpha,\beta=1}^{\infty}\Big\{\delta_{\alpha\beta}+\sum_{n=1}^{\infty}\lambda^n k_{\alpha\beta}^{(n)}\Big\}\bar{X}_\alpha X_\beta = \sum_{\gamma=1}^{\infty}\frac{\bar{Y}_\gamma Y_\gamma}{1-\lambda/\lambda_\gamma}.$$

If all the eigen-values λ_γ are known to have the same sign as λ, the coefficient $1/(1+|\lambda|b)$ in the first term of (22) can be replaced by 1. The bound of the Hermitian matrix $\Big[\delta_{\alpha\beta}+\sum_{n=1}^{\infty}\lambda^n k_{\alpha\beta}^{(n)}\Big]$ is

$$\frac{1}{1-|\lambda/\lambda_1|}=\frac{1}{1-|\lambda|b}. \tag{23}$$

This can be used to examine the order of magnitude of the x_α given by (20). For

$$\begin{aligned} x_\alpha &= c_\alpha+\sum_{\beta=1}^{\infty}c_\beta\sum_{n=1}^{\infty}\lambda^n k_{\alpha\beta}^{(n)} \\ &= c_\alpha+\lambda\sum_{\beta,\gamma=1}^{\infty}k_{\alpha\gamma}\Big\{\delta_{\gamma\beta}+\sum_{n=1}^{\infty}\lambda^n k_{\gamma\beta}^{(n)}\Big\}c_\beta, \end{aligned}$$

so that

$$|x_\alpha| \leqslant |c_\alpha|+\frac{|\lambda|\Big\{\sum_{\gamma=1}^{\infty}|k_{\alpha\gamma}|^2\Big\}^{\frac12}\Big\{\sum_{\beta=1}^{\infty}|c_\beta|^2\Big\}^{\frac12}}{1-|\lambda|b}. \tag{24}$$

3. Reduction of the infinite system of equations to a finite system

The system of equations

$$x_\alpha-\lambda\sum_{\beta=1}^{\infty}k_{\alpha\beta}x_\beta = 0 \quad (\alpha = 1,2,\ldots) \tag{1}$$

is readily reduced to

$$x_\alpha-\sum_{\beta=1}^{m}r_{\alpha\beta}x_\beta-\lambda\sum_{\beta=1}^{m}k_{\alpha\beta}x_\beta = 0 \quad (\alpha = 1,\ldots,m) \tag{3}$$

by solving for $x_{m+1}, x_{m+2},\ldots$ in terms of $x_1,\ldots, x_m$ by iteration and then eliminating. To do this, equations (1) are separated into two groups

$$x_\alpha-\lambda\sum_{\beta=1}^{m}k_{\alpha\beta}x_\beta-\lambda\sum_{\beta=m+1}^{\infty}k_{\alpha\beta}x_\beta = 0 \quad (\alpha = 1,\ldots,m), \tag{25}$$

$$x_\alpha-\lambda\sum_{\beta=m+1}^{\infty}k_{\alpha\beta}x_\beta = \lambda\sum_{\beta=1}^{m}k_{\alpha\beta}x_\beta \quad (\alpha = m+1, m+2,\ldots). \tag{26}$$

To avoid confusion the last group will be written as

$$x_\alpha-\lambda\sum_{\beta=m+1}^{\infty}K_{\alpha\beta}x_\beta = \lambda\sum_{\beta=1}^{m}k_{\alpha\beta}x_\beta \quad (\alpha = m+1, m+2,\ldots), \tag{27}$$

where
$$K_{\alpha\beta}\begin{cases} = 0 & (\alpha,\beta \not> m),\\ = k_{\alpha\beta} & (\alpha,\beta > m).\end{cases}$$

Equations (27) can be solved by iteration, giving

$$x_\alpha = \sum_{\gamma=m+1}^{\infty}\Big\{\delta_{\alpha\gamma}+\sum_{n=1}^{\infty}\lambda^n K_{\alpha\gamma}^{(n)}\Big\}\lambda\sum_{\beta=1}^{m}k_{\gamma\beta}x_\beta \quad (\alpha = m+1, m+2,...), \tag{28}$$

provided that b_m, the bound of $[K_{\alpha\beta}]$, is less than $|1/\lambda|$, i.e. if

$$|\lambda|b_m < 1. \tag{29}$$

From (6) this is satisfied if m is sufficiently large. On intuitive grounds it might be expected that b_m is an approximation to $|1/\lambda_{m+1}|$ and therefore that $|\lambda|b_m < 1$ for $|\lambda| \leqslant |\lambda_m|$. It will be assumed that (29) holds.

If the unknowns $x_{m+1}, x_{m+2},...$ given by (28) are substituted in (25), there results

$$x_\alpha - \lambda\sum_{\beta=1}^{m}k_{\alpha\beta}x_\beta - \lambda^2\sum_{\gamma,\epsilon=m+1}^{\infty}k_{\alpha\gamma}\Big\{\delta_{\gamma\epsilon}+\sum_{n=1}^{\infty}\lambda^n K_{\gamma\epsilon}^{(n)}\Big\}\sum_{\beta=1}^{m}k_{\epsilon\beta}x_\beta = 0$$
$$(\alpha = 1,..., m). \tag{30}$$

This is of the form (3) with

$$r_{\alpha\beta} = \lambda^2\sum_{\gamma,\epsilon=m+1}^{\infty}\Big\{\delta_{\gamma\epsilon}+\sum_{n=1}^{\infty}\lambda^n K_{\gamma\epsilon}^{(n)}\Big\}k_{\alpha\gamma}k_{\epsilon\beta}. \tag{31}$$

Obviously $r_{\alpha\beta}$ is Hermitian for real λ. Equations (30) together with (28) are completely equivalent to equations (1), provided that the convergence condition (29) is satisfied.

Before passing to the next section it should be observed that the result of applying (24) to (28) is

$$|x_\alpha| \leqslant \Big|\lambda\sum_{\beta=1}^{m}k_{\alpha\beta}x_\beta\Big| + \frac{|\lambda|\Big\{\sum_{\gamma=m+1}^{\infty}|K_{\alpha\gamma}|^2\Big\}^{\frac{1}{2}}\Big\{\sum_{\gamma=m+1}^{\infty}\Big|\sum_{\beta=1}^{m}k_{\gamma\beta}x_\beta\Big|^2\Big\}^{\frac{1}{2}}}{1-|\lambda|b_m}$$
$$\leqslant |\lambda|\Big\{\sum_{\beta=1}^{m}|x_\beta|^2\Big\}^{\frac{1}{2}}\left[\Big\{\sum_{\beta=1}^{m}|k_{\alpha\beta}|^2\Big\}^{\frac{1}{2}} + \frac{\Big\{\sum_{\gamma=m+1}^{\infty}|K_{\alpha\gamma}|^2\Big\}^{\frac{1}{2}}\Big\{\sum_{\gamma=m+1}^{\infty}\sum_{\beta=1}^{m}|k_{\gamma\beta}|^2\Big\}^{\frac{1}{2}}}{1-|\lambda|b_m}\right]$$
$$(\alpha = m+1, m+2,...), \tag{32}$$

where Schwarz's inequality $\Big|\sum_\alpha a_\alpha b_\alpha\Big| \leqslant \Big\{\sum_\alpha |a_\alpha|^2\Big\}^{\frac{1}{2}}\Big\{\sum_\alpha |b_\alpha|^2\Big\}^{\frac{1}{2}}$ has been used in simplifying the result.

4. Comparisón of the curtailed and exact systems of equations

For the curtailed system of equations

$$x_\alpha - \lambda \sum_{\beta=1}^{m} k_{\alpha\beta} x_\beta = 0 \quad (\alpha = 1,\dots, m) \tag{2}$$

in all the results given by equations (7)–(14) summations from 1 to m replace summations from 1 to ∞. To indicate that the curtailed system is being considered the symbols will be modified by the addition of a superscript (m). The m eigen-values $\lambda_1^{(m)},\dots, \lambda_m^{(m)}$ of (2) are found from the equation

$$\det(\delta_{\alpha\beta} - \lambda k_{\alpha\beta})^{(m)} = 0, \tag{33}$$

which is the condition that (2) has a solution other than the null solution $x_\alpha = 0$ $(\alpha = 1,\dots, m)$. If $w_{\alpha\gamma}^{(m)}$ is the normalized eigen-vector corresponding to $\lambda_\gamma^{(m)}$, equations (2) can be reduced to the form

$$y_\gamma - (\lambda/\lambda_\gamma^{(m)}) y_\gamma = 0 \quad (\gamma = 1,\dots, m) \tag{34}$$

by the unitary transformation

$$\left.\begin{aligned} y_\gamma &= \sum_{\alpha=1}^{m} \bar{w}_{\alpha\gamma}^{(m)} x_\alpha \\ x_\alpha &= \sum_{\gamma=1}^{m} w_{\alpha\gamma}^{(m)} y_\gamma. \end{aligned}\right\} \tag{35}$$

For the exact system (3) the results are not as simple. The determinantal equation

$$\det(\delta_{\alpha\beta} - \lambda k_{\alpha\beta} - r_{\alpha\beta}) = 0 \tag{36}$$

is the condition that equations (3) and hence equations (1) possess a non-trivial solution. It will be shown that (36) determines m eigen-values $\lambda_1,\dots, \lambda_m$ to which $\lambda_1^{(m)},\dots, \lambda_m^{(m)}$ are approximations provided that the $r_{\alpha\beta}$ are small. The values of the x_α $(\alpha = 1,\dots, m)$ will be proportional to $w_{1\gamma},\dots, w_{m\gamma}$, the first m components of the eigen-vector $w_{\alpha\gamma}$ of (1). The remaining components are given by (28). In general the vectors $\{w_{1\gamma},\dots, w_{m\gamma}, 0,\dots, 0,\dots\}$ will not be orthogonal.

It is convenient to transform equations (3) by (35) rather than to discuss them as they stand. This gives

$$y_\alpha(1 - \lambda/\lambda_\alpha^{(m)}) - \sum_{\beta=1}^{m} r'_{\alpha\beta} y_\beta \doteq 0 \quad (\alpha = 1,\dots, m), \tag{37}$$

where

$$r'_{\alpha\beta} = \sum_{\gamma,\epsilon=1}^{m} \bar{w}_{\gamma\alpha}^{(m)} w_{\epsilon\beta}^{(m)} r_{\gamma\epsilon}$$

$$= \lambda^2 \sum_{\eta,\theta=m+1}^{\infty} \left\{\delta_{\eta\theta} + \sum_{n=1}^{\infty} \lambda^n K_{\eta\theta}^{(n)}\right\} \sum_{\gamma,\epsilon=1}^{m} k_{\gamma\theta} k_{\eta\epsilon} \bar{w}_{\gamma\alpha}^{(m)} w_{\epsilon\beta}^{(m)}$$

$$= \sum_{\eta,\theta=m+1}^{\infty} \left\{\delta_{\eta\theta} + \sum_{n=1}^{\infty} \lambda^n K_{\eta\theta}^{(n)}\right\} r_\theta^{(\alpha)} \bar{r}_\eta^{(\beta)} \tag{38}$$

and

$$r_\theta^{(\alpha)} = \lambda \sum_{\gamma=1}^{m} k_{\gamma\theta} \bar{w}_{\gamma\alpha}^{(m)}. \tag{39}$$

Like $[r_{\alpha\beta}]$, $[r'_{\alpha\beta}]$ is Hermitian. By applying the unitary transformation which reduces $[K_{\alpha\beta}]$ to diagonal form (compare the derivations of equations (21), (22)), $r'_{\alpha\beta}$ can be expressed as

$$r'_{\alpha\beta} = \sum_{\gamma=m+1}^{\infty} \frac{s_\gamma^{(\alpha)} \bar{s}_\gamma^{(\beta)}}{1-\lambda/\mu_\gamma} = \sum_{\gamma=m+1}^{\infty} \frac{s_\gamma^{(\alpha)}}{(1-\lambda/\mu_\gamma)^{\frac{1}{2}}} \frac{\bar{s}_\gamma^{(\beta)}}{(1-\lambda/\mu_\gamma)^{\frac{1}{2}}},$$

where μ_{m+1}, μ_{m+2},... are the eigen-values corresponding to $[K_{\alpha\beta}]$, $|\lambda/\mu_\gamma| < 1$ $(\gamma = m+1, m+2,...)$ since $b_m = |1/\mu_{m+1}|$, and

$$\sum_{\gamma=m+1}^{\infty} |s_\gamma^{(\alpha)}|^2 = \sum_{\theta=m+1}^{\infty} |r_\theta^{(\alpha)}|^2.$$

It is easy to see that

$$|r'_{\alpha\beta}|^2 \leqslant r'_{\alpha\alpha} r'_{\beta\beta}, \tag{40}$$

with the equality sign holding if and only if $x_\gamma^{(\alpha)} = kx_\gamma^{(\beta)}$ for each γ, where k is independent of γ. It is also obvious that $r'_{\alpha\alpha}$ is a continuous function, zero for $\lambda = 0$, and increasing as $|\lambda|$ increases, since $s_\gamma^{(\alpha)}$ contains a factor λ.

Applying the result (22) to (38) gives

$$\frac{\sum_{\gamma=m+1}^{\infty} |r_\gamma^{(\alpha)}|^2}{1+|\lambda|b_m} \leqslant r'_{\alpha\alpha} \leqslant \frac{\sum_{\gamma=m+1}^{\infty} |r_\gamma^{(\alpha)}|^2}{1-|\lambda|b_m}. \tag{41}$$

A rough upper bound for $r'_{\alpha\alpha}$ is given by

$$r' = \sum_{\alpha=1}^{m} r'_{\alpha\alpha} = \lambda^2 \sum_{\eta,\theta=m+1}^{\infty} \left\{\delta_{\eta\theta} + \sum_{n=1}^{\infty} \lambda^n K_{\eta\theta}^{(n)}\right\} \sum_{\gamma=1}^{m} k_{\gamma\theta} k_{\eta\gamma}$$

$$= \lambda^2 \sum_{\gamma=1}^{m} \sum_{\eta,\theta=m+1}^{\infty} \left\{\delta_{\eta\theta} + \sum_{n=1}^{\infty} \lambda^n K_{\eta\theta}^{(n)}\right\} k_{\gamma\theta} \bar{k}_{\gamma\eta}.$$

For r' we have

$$\frac{\lambda^2 \sum_{\gamma=1}^{m} \sum_{\eta=m+1}^{\infty} |k_{\gamma\eta}|^2}{1+|\lambda|b_m} \leqslant r' \leqslant \frac{\lambda^2 \sum_{\gamma=1}^{m} \sum_{\eta=m+1}^{\infty} |k_{\gamma\eta}|^2}{1-|\lambda|b_m}. \tag{42}$$

The assumption that $\sum_{\alpha=1}^{m}\sum_{\beta=m+1}^{\infty}|k_{\alpha\beta}|^2$ tends to zero with increasing m ensures that r', and hence $r'_{\alpha\alpha}$, also tend to zero. If the conditions of the following theorem are satisfied for all m sufficiently great, it is necessary that $r'_{\alpha\alpha}\to 0$ in order that $\lambda_\alpha^{(m)}\to\lambda_\alpha$ as $m\to\infty$.

THEOREM. *The eigen-value λ_α ($\alpha = 1,\dots,m$) of the system* (1) *will lie in the range*

$$1-(r'_{\alpha\alpha})_{\lambda=\lambda''}\left\{1+\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\}_{\lambda=\lambda_\alpha^{(m)}}$$
$$<\lambda_\alpha/\lambda_\alpha^{(m)}<1-(r'_{\alpha\alpha})_{\lambda=\lambda'}\left\{1-\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\}_{\lambda=\lambda_\alpha^{(m)}} \tag{43}$$

where λ', λ'' are any numbers such that

$$1-(r'_{\alpha\alpha})_{\lambda=\lambda_\alpha^{(m)}}\left\{1+\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\}_{\lambda=\lambda_\alpha^{(m)}}\leqslant\lambda'/\lambda_\alpha^{(m)}\leqslant 1-(r'_{\alpha\alpha})_{\lambda=\lambda'}\left\{1+\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\},$$

$$1-(r'_{\alpha\alpha})_{\lambda=\lambda'}\left\{1-\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\}_{\lambda=\lambda_\alpha^{(m)}}\leqslant\lambda''/\lambda_\alpha^{(m)}\leqslant 1,$$

and t_α is any constant such that

$$0<t_\alpha\leqslant\min|1-\lambda/\lambda_\beta^{(m)}|\quad(\beta\neq\alpha;\,1-2(r'_{\alpha\alpha})_{\lambda=\lambda_\alpha^{(m)}}\leqslant\lambda/\lambda_\alpha^{(m)}\leqslant 1)$$

provided that

$$(r'/t_\alpha)_{\lambda=\lambda_\alpha^{(m)}}\leqslant\tfrac{1}{2}.$$

If λ_α is the smallest positive eigen-value or the smallest in magnitude of the negative eigen-values, so that

$$1-\lambda/\lambda_\beta^{(m)}\quad(\beta\neq\alpha;\;1-2(r'_{\alpha\alpha})_{\lambda=\lambda_\alpha^{(m)}}\leqslant\lambda/\lambda_\alpha^{(m)}\leqslant 1)$$

is necessarily positive, (43) *can be replaced by the smaller range*

$$1-(r'_{\alpha\alpha})_{\lambda=\lambda''}\left\{1+\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\}_{\lambda=\lambda_\alpha^{(m)}}<\lambda_\alpha/\lambda_\alpha^{(m)}<1-(r'_{\alpha\alpha})_{\lambda=\lambda'} \tag{44}$$

where λ', λ'' satisfy the slightly modified conditions

$$1-(r'_{\alpha\alpha})_{\lambda=\lambda_\alpha^{(m)}}\left\{1+\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\}_{\lambda=\lambda_\alpha^{(m)}}$$
$$\leqslant\lambda'/\lambda_\alpha^{(m)}\leqslant 1-(r'_{\alpha\alpha})_{\lambda=\lambda'}\left\{1+\frac{r'/t_\alpha}{1-r'/t_\alpha}\right\}_{\lambda=\lambda_\alpha^{(m)}},$$

$$1-(r'_{\alpha\alpha})_{\lambda=\lambda'}\leqslant\lambda''/\lambda_\alpha^{(m)}\leqslant 1,$$

again provided that $$(r'/t_\alpha)_{\lambda=\lambda_\alpha^{(m)}} \leqslant \tfrac{1}{2}.$$

This theorem will now be proved taking, for definiteness, $\alpha = 1$. Equations (37) will be separated into

$$y_1(1-\lambda/\lambda_1^{(m)}-r'_{11})-\sum_{\beta=2}^{m} r'_{1\beta}y_\beta = 0 \tag{45}$$

and $$y_\alpha(1-\lambda/\lambda_\alpha^{(m)})-\sum_{\beta=2}^{m} r'_{\alpha\beta}y_\beta = r'_{\alpha 1}y_1 \quad (\alpha > 1). \tag{46}$$

Dividing (46) by $(1-\lambda/\lambda_\alpha^{(m)})$ gives

$$y_\alpha-\sum_{\beta=2}^{m}\frac{r'_{\alpha\beta}y_\beta}{1-\lambda/\lambda_\alpha^{(m)}} = \frac{r'_{\alpha 1}y_1}{1-\lambda/\lambda_\alpha^{(m)}} \quad (\alpha > 1), \tag{47}$$

and this system can be solved for y_α by iteration. Substituting the result in (45) reduces this to the form

$$y_1(1-\lambda/\lambda_1^{(m)}-r'_{11}-e_{11}) = 0, \tag{48}$$

where $$e_{11} = \sum_{\alpha=2}^{m}\frac{r'_{1\alpha}r'_{\alpha 1}}{1-\lambda/\lambda_\alpha^{(m)}}+\sum_{\alpha,\beta=2}^{m}\frac{r'_{1\alpha}r'_{\alpha\beta}r'_{\beta 1}}{(1-\lambda/\lambda_\alpha^{(m)})(1-\lambda/\lambda_\beta^{(m)})}+\dots.$$

It is easy to verify that e_{11} is continuous and real for real λ. The condition that (47) can be solved for y_α by iteration is that

$$b\left[\frac{r'_{\alpha\beta}}{1-\lambda/\lambda_\alpha^{(m)}}\right] < 1.$$

Now $$b\left[\frac{r'_{\alpha\beta}}{1-\lambda/\lambda_\alpha^{(m)}}\right] \leqslant \left\{\sum_{\alpha,\beta=2}^{m}\left|\frac{r'_{\alpha\beta}}{1-\lambda/\lambda_\alpha^{(m)}}\right|^2\right\}^{\frac{1}{2}}.$$

Let t be a positive constant such that

$$t \leqslant |1-\lambda/\lambda_\alpha^{(m)}| \quad (\alpha > 1;\ 1-2(r'_{11})_{\lambda=\lambda_1^{(m)}} \leqslant \lambda/\lambda_1^{(m)} \leqslant 1).$$

Then $$\left\{\sum_{\alpha,\beta=2}^{m}\left|\frac{r'_{\alpha\beta}}{1-\lambda/\lambda_\alpha^{(m)}}\right|^2\right\}^{\frac{1}{2}} \leqslant \frac{1}{t}\left\{\sum_{\alpha,\beta=2}^{m}|r'_{\alpha\beta}|^2\right\}^{\frac{1}{2}}$$

$$\leqslant \frac{1}{t}\left\{\sum_{\alpha,\beta=2}^{m} r'_{\alpha\alpha}r'_{\beta\beta}\right\}^{\frac{1}{2}}$$

$$= \frac{r'-r'_{11}}{t}.$$

If $r'-r'_{11} < t$, the formal work used in deriving (48) is certainly valid.

Bounds will now be sought for e_{11}. We have

$$|e_{11}| \leqslant \sum_{\alpha=2}^{m} \frac{|r'_{\alpha 1}|^2}{|1-\lambda/\lambda_\alpha^{(m)}|} + \sum_{\alpha,\beta=2}^{m} \frac{|r'_{1\alpha}|\,|r'_{\alpha\beta}|\,|r'_{\beta 1}|}{|1-\lambda/\lambda_\alpha^{(m)}|\,|1-\lambda/\lambda_\beta^{(m)}|} + \dots$$

$$\leqslant r'_{11}\left\{\sum_{\alpha=2}^{m} \frac{r'_{\alpha\alpha}}{t} + \sum_{\alpha,\beta=2}^{m} \frac{r'_{\alpha\alpha} r'_{\beta\beta}}{t^2} + \dots\right\}$$

$$= r'_{11}\, \frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)$$

$$\leqslant r'_{11}\left\{\frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)\right\}_{\lambda=\lambda_1^{(m)}}, \text{ where } 1-2(r'_{11})_{\lambda=\lambda_1^{(m)}} \leqslant \lambda/\lambda_1^{(m)} \leqslant 1 \qquad (49)$$

since $r'-r'_{11}$ increases as $\lambda/\lambda_1^{(m)}$ increases.

If $$(r'-r'_{11})_{\lambda=\lambda_1^{(m)}} \leqslant \tfrac{1}{2}t,$$

then $$|e_{11}| \leqslant r'_{11}, \quad \text{where } 1-2(r'_{11})_{\lambda=\lambda_1^{(m)}} \leqslant \lambda/\lambda_1^{(m)} \leqslant 1.$$

It will be assumed that this condition holds.

If λ, λ', λ'' are such that

$$1-2(r'_{11})_{\lambda=\lambda_1^{(m)}} \leqslant \lambda'/\lambda_1^{(m)} \leqslant \lambda/\lambda_1^{(m)} \leqslant \lambda''/\lambda_1^{(m)} \leqslant 1,$$

then $$1-(r'_{11})_{\lambda=\lambda'}\left\{1+\frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)\right\}_{\lambda=\lambda_1^{(m)}}$$

$$\leqslant (1-r'_{11}-e_{11})_\lambda$$

$$\leqslant 1-(r'_{11})_{\lambda=\lambda'}\left\{1-\frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)\right\}_{\lambda=\lambda_1^{(m)}}.$$

Further, if

$$\lambda'/\lambda_1^{(m)} \leqslant 1-(r'_{11})_{\lambda=\lambda'}\left\{1+\frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)\right\}_{\lambda=\lambda_1^{(m)}}$$

and $$\lambda''/\lambda_1^{(m)} \geqslant 1-(r'_{11})_{\lambda=\lambda'}\left\{1-\frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)\right\}_{\lambda=\lambda_1^{(m)}},$$

then $(1-r'_{11}-e_{11}-\lambda/\lambda_1^{(m)})$ is not positive for $\lambda = \lambda'$ and not negative for $\lambda = \lambda''$. From continuity there is a zero $\lambda = \lambda_1$ of

$$(1-r'_{11}-e_{11}-\lambda/\lambda_1^{(m)})$$

in the range

$$1-(r'_{11})_{\lambda=\lambda'}\left\{1+\frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)\right\}_{\lambda=\lambda_1^{(m)}}$$

$$\leqslant \lambda/\lambda_1^{(m)} \leqslant 1-(r'_{11})_{\lambda=\lambda'}\left\{1-\frac{r'-r'_{11}}{t} \Big/ \left(1-\frac{r'-r'_{11}}{t}\right)\right\}_{\lambda=\lambda_1^{(m)}}. \qquad (50)$$

The effect of replacing $(r'-r'_{11})/t$ by r'/t is to increase the range and gives

$$1-(r'_{11})_{\lambda=\lambda''}\left\{1+\frac{r'/t}{1-r'/t}\right\}_{\lambda=\lambda_1^{(m)}} \leqslant \lambda/\lambda_1^{(m)} \leqslant 1-(r'_{11})_{\lambda=\lambda'}\left\{1-\frac{r'/t}{1-r'/t}\right\}_{\lambda=\lambda_1^{(m)}} \tag{43}$$

where λ', λ'' are any numbers such that

$$1-2(r'_{11})_{\lambda=\lambda_1^{(m)}} \leqslant \lambda'/\lambda_1^{(m)} \leqslant 1-(r'_{11})_{\lambda=\lambda''}\left\{1+\frac{r'/t}{1-r'/t}\right\}_{\lambda=\lambda_1^{(m)}},$$

$$1-(r'_{11})_{\lambda=\lambda'}\left\{1-\frac{r'/t}{1-r'/t}\right\}_{\lambda=\lambda_1^{(m)}} \leqslant \lambda'/\lambda_1^{(m)} \leqslant 1.$$

By taking $\lambda''=\lambda_1^{(m)}$, it is obvious that the condition

$$1-(r'_{11})_{\lambda=\lambda_1^{(m)}}\left\{1+\frac{r'/t}{1-r'/t}\right\}_{\lambda=\lambda_1^{(m)}} \leqslant \lambda'/\lambda_1^{(m)}$$

can be added. The first part of the theorem has thus been proved. The second part is deduced in an exactly similar manner using instead of (49) the inequalities

$$0 \leqslant e_{11} \leqslant r'_{11}\frac{r'-r'_{11}}{t}\bigg/\left(1-\frac{r'-r'_{11}}{t}\right), \tag{51}$$

which hold if $(1-\lambda/\lambda_\alpha^{(m)})$ is positive for each α. To prove (51) it has to be shown that e_{11} is positive. Now

$$e_{11} = \sum_{\alpha=2}^{m}\frac{|r'_{1\alpha}|^2}{1-\lambda/\lambda_\alpha^{(m)}}+\sum_{\alpha,\beta=2}^{m}\frac{r'_{1\alpha}r'_{\alpha\beta}r'_{\beta1}}{(1-\lambda/\lambda_\alpha^{(m)})(1-\lambda/\lambda_\beta^{(m)})}+\dots,$$

and the sum of the terms on the right (excluding the first) is less in magnitude than

$$\sum_{\alpha=2}^{m}\frac{|r'_{1\alpha}|(r'_{\alpha\alpha})^{\frac12}}{1-\lambda/\lambda_\alpha^{(m)}}\sum_{\beta=2}^{m}\frac{(r'_{\beta\beta})^{\frac12}|r'_{\beta1}|}{1-\lambda/\lambda_\beta^{(m)}}\left\{1+\sum_{\gamma=2}^{m}\frac{r'_{\gamma\gamma}}{1-\lambda/\lambda_\gamma^{(m)}}+\dots\right\}$$

$$\leqslant \left\{\sum_{\alpha=2}^{m}\frac{|r'_{1\alpha}|^2}{1-\lambda/\lambda_\alpha^{(m)}}\sum_{\gamma=2}^{m}\frac{r'_{\gamma\gamma}}{1-\lambda/\lambda_\gamma^{(m)}}\right\}^{\frac12}\left\{\sum_{\beta=2}^{m}\frac{|r'_{\beta1}|^2}{1-\lambda/\lambda_\beta^{(m)}}\sum_{\epsilon=2}^{m}\frac{r'_{\epsilon\epsilon}}{1-\lambda/\lambda_\epsilon^{(m)}}\right\}^{\frac12}\times$$

$$\times\frac{1}{1-(r'-r'_{11})/t}$$

$$= \sum_{\alpha=2}^{m}\frac{|r'_{1\alpha}|^2}{1-\lambda/\lambda_\alpha^{(m)}}\,\frac{(r'-r'_{11})/t}{1-(r'-r'_{11})/t}$$

$$\leqslant \sum_{\alpha=2}^{m}\frac{|r'_{1\alpha}|^2}{1-\lambda/\lambda_\alpha^{(m)}}$$

if $r'-r'_{11} \leqslant \frac12 t$.

Since the first term in the expression for e_{11} is positive and the remaining terms have a sum less in magnitude than the first term, e_{11} is certainly positive. Having established (51) it is then simple to derive (44), i.e. the second part of the theorem.

If the bounds are to be of practical value r'/t must be small, preferably much less than $\frac{1}{2}$. If r'/t is not small, a more careful discussion is needed. This case corresponds to two or more of the $\lambda_\alpha^{(m)}$ being nearly coincident. Suppose that $\lambda_1^{(m)} \approx \lambda_2^{(m)}$, so that t is small. Then $y_3,..., y_m$ are eliminated as before, yielding the equations

$$\left.\begin{aligned} y_1(1-\lambda/\lambda_1^{(m)}-r'_{11}-e_{11})-y_2(r_{12}+e_{12}) &= 0 \\ -y_1(r_{21}+e_{21})+y_2(1-\lambda/\lambda_2^{(m)}-r'_{22}-e_{22}) &= 0, \end{aligned}\right\} \tag{52}$$

where

$$|e_{11}| \leqslant \frac{r'_{11}}{1-(r'-r'_{11}-r'_{22})/t}\,\frac{r'-r'_{11}-r'_{22}}{t},$$

$$|e_{22}| \leqslant \frac{r'_{22}}{1-(r'-r'_{11}-r'_{22})/t}\,\frac{r'-r'_{11}-r'_{22}}{t},$$

$$|e_{12}| = |e_{21}| \leqslant \frac{(r'_{11})^{\frac{1}{2}}(r'_{22})^{\frac{1}{2}}}{1-(r'-r'_{11}-r'_{22})/t}\,\frac{r'-r'_{11}-r'_{22}}{t},$$

and t has the modified definition

$$t \leqslant |1-\lambda/\lambda_\alpha^{(m)}| \quad (\alpha > 2;\ 1-2(r'_{11}+r'_{22})_{\lambda=\lambda_1^{(m)}} \leqslant \lambda/\lambda_1^{(m)},\ \lambda/\lambda_2^{(m)} \leqslant 1).$$

Equations (52) will not be discussed in detail. However, neglecting e_{11}, e_{12}, e_{21}, e_{22} and using the result $|r'_{12}|^2 \leqslant r'_{11}r'_{22}$, it is easy to show that there are zeros λ_1, λ_2 of

$$\begin{vmatrix} 1-\lambda/\lambda_1^{(m)}-r'_{11} & -r'_{12} \\ -r'_{21} & 1-\lambda/\lambda_2^{(m)}-r'_{22} \end{vmatrix}$$

in the ranges $\quad 1-r'_{11} \geqslant \lambda_1/\lambda_1^{(m)} \geqslant 1-r'_{11}-r'_{22}$

and $\quad 1 \geqslant \lambda_2/\lambda_2^{(m)} \geqslant 1-r'_{22}$. (53)

5. Extension of the theory

There is no particular difficulty in extending the theory to systems of the form

$$x_\alpha + \sum_{\beta=1}^{\infty} h_{\alpha\beta}x_\beta - \lambda\sum_{\beta=1}^{\infty} k_{\alpha\beta}x_\beta = 0 \quad (\alpha = 1,2,...). \tag{54}$$

The same conditions will be placed on the $h_{\alpha\beta}$ as on the $k_{\alpha\beta}$, namely that $[h_{\alpha\beta}]$ is Hermitian, completely continuous, and that

$$\sum_{\alpha=1}^{m}\sum_{\beta=m+1}^{\infty} |h_{\alpha\beta}|^2$$

tends to zero with increasing m. The further assumption that $[\delta_{\alpha\beta}+h_{\alpha\beta}]$ is positive definite will be added.

By a unitary transformation

$$\left.\begin{aligned} y_\gamma &= \sum_{\alpha=1}^{\infty} \bar{u}_{\alpha\gamma} x_\alpha \\ x_\alpha &= \sum_{\gamma=1}^{\infty} u_{\alpha\gamma} y_\gamma \end{aligned}\right\} \tag{55}$$

the equations (54) can be reduced to the form

$$y_\alpha(1+h_\alpha)-\lambda\sum_{\beta=1}^{\infty} k'_{\alpha\beta} y_\beta = 0 \quad (\alpha = 1,2,...). \tag{56}$$

Writing $\quad y_\alpha(1+h_\alpha)^{\frac{1}{2}} = y'_\alpha, \qquad \dfrac{k'_{\alpha\beta}}{(1+h_\alpha)^{\frac{1}{2}}(1+h_\beta)^{\frac{1}{2}}} = k''_{\alpha\beta} \qquad (57)$

brings (56) to the form

$$y'_\alpha-\lambda\sum_{\beta=1}^{\infty} k''_{\alpha\beta} y'_\beta = 0 \quad (\alpha = 1,2,...). \tag{58}$$

Since $[\delta_{\alpha\beta}+h_{\alpha\beta}]$ is positive definite the $(1+h_\alpha)$ are positive and $k''_{\alpha\beta}$ is Hermitian. The eigen-values of (58) are therefore real. Finally the unitary transformation

$$\left.\begin{aligned} z_\gamma &= \sum_{\alpha=1}^{\infty} \bar{w}_{\alpha\gamma} y'_\alpha \\ y'_\alpha &= \sum_{\gamma=1}^{\infty} w_{\alpha\gamma} z_\gamma \end{aligned}\right\} \tag{59}$$

reduces (58) to $\quad z_\gamma(1-\lambda/\lambda_\gamma) = 0 \quad (\gamma = 1,2,...). \qquad (60)$

The eigen-vector $x_{\alpha\gamma}$ corresponding to λ_γ is given by

$$x_{\alpha\gamma} = \sum_{\beta=1}^{\infty} \frac{u_{\alpha\beta} w_{\beta\gamma}}{(1+h_\beta)^{\frac{1}{2}}} z_\gamma. \tag{61}$$

Exactly the same process as that used in § 3 can be applied to (54). This gives

$$x_\alpha-\sum_{\beta=1}^{m} r_{\alpha\beta} x_\beta+\sum_{\beta=1}^{m} h_{\alpha\beta} x_\beta-\lambda\sum_{\beta=1}^{m} k_{\alpha\beta} x_\beta = 0 \quad (\alpha = 1,..., m), \tag{62}$$

where

$$r_{\alpha\beta} = \sum_{\gamma,\epsilon=m+1}^{\infty}\Big\{\delta_{\gamma\epsilon}+\sum_{n=1}^{\infty}\lambda^n K_{\gamma\epsilon}^{(n)}\Big\}(\lambda k_{\alpha\gamma}-h_{\alpha\gamma})(\lambda k_{\epsilon\beta}-h_{\epsilon\beta}), \tag{63}$$

$$K_{\gamma\epsilon} = \begin{cases} k_{\gamma\epsilon}-(1/\lambda)h_{\gamma\epsilon} & (\gamma,\epsilon > m), \\ 0 & (\gamma,\epsilon \not> m), \end{cases}$$

and the $K_{\gamma\epsilon}^{(n)}$ are defined as before. The properties

$$|r_{\alpha\beta}|^2 \leqslant r_{\alpha\alpha} r_{\beta\beta},$$

$$\frac{\sum\limits_{\gamma=m+1}^{\infty} |\lambda k_{\alpha\gamma}-h_{\alpha\gamma}|^2}{1+|\lambda|b_m} \leqslant r_{\alpha\alpha} \leqslant \frac{\sum\limits_{\gamma=m+1}^{\infty} |\lambda k_{\alpha\gamma}-h_{\alpha\gamma}|^2}{1-|\lambda|b_m}$$

still hold. Minkowski's inequality then gives

$$r_{\alpha\alpha} \leqslant \frac{\left\{\lambda\left(\sum\limits_{\gamma=m+1}^{\infty} |k_{\alpha\gamma}|^2\right)^{\frac{1}{2}}+\left(\sum\limits_{\gamma=m+1}^{\infty} |h_{\alpha\gamma}|^2\right)^{\frac{1}{2}}\right\}^2}{1-|\lambda|b_m}. \tag{64}$$

Applying the transformations for the curtailed system analogous to (55), (57), (59) reduces (62) to

$$z_\alpha(1-\lambda/\lambda_\alpha^{(m)})-\sum_{\beta=1}^{m} r'_{\alpha\beta} z_\beta = 0 \quad (\alpha = 1,\dots, m), \tag{65}$$

where
$$r'_{\alpha\beta} = \sum_{\gamma,\epsilon,\eta,\theta=1}^{m} \frac{\bar{u}_{\gamma\eta}^{(m)}\, \bar{w}_{\eta\alpha}^{(m)}}{(1+h_\eta^{(m)})^{\frac{1}{2}}} r_{\gamma\epsilon} \frac{u_{\epsilon\theta}^{(m)} u_{\theta\beta}^{(m)}}{(1+h_\theta^{(m)})^{\frac{1}{2}}}. \tag{66}$$

By using (61), $r'_{\alpha\beta}$ can be rewritten as

$$r'_{\alpha\beta} = \sum_{\gamma,\epsilon=1}^{m} r_{\gamma\epsilon}\left(\frac{\bar{x}_{\gamma\alpha}^{(m)}}{\bar{z}_\alpha^{(m)}}\right)\left(\frac{x_{\epsilon\beta}^{(m)}}{z_\beta^{(m)}}\right), \tag{67}$$

and finally it can be brought to the same form as (38) (which it in fact includes as a special case),

$$r'_{\alpha\beta} = \sum_{\eta,\theta=m+1}^{\infty} \left\{\delta_{\eta\theta}+\sum_{n=1}^{\infty} \lambda^n K_{\eta\theta}^{(n)}\right\} r_\theta^{(\alpha)} \bar{r}_\eta^{(\beta)}, \tag{68}$$

where
$$r_\theta^{(\alpha)} = \sum_{\gamma=1}^{m} (\lambda k_{\gamma\theta}-h_{\gamma\theta}) \frac{\bar{x}_{\gamma\alpha}^{(m)}}{\bar{z}_\alpha^{(m)}}. \tag{69}$$

For $r'_{\alpha\alpha}$ the inequalities

$$\frac{\sum\limits_{\theta=m+1}^{\infty} |r_\theta^{(\alpha)}|^2}{1+|\lambda|b_m} \leqslant r'_{\alpha\alpha} \leqslant \frac{\sum\limits_{\theta=m+1}^{\infty} |r_\theta^{(\alpha)}|^2}{1-|\lambda|b_m} \tag{70}$$

hold.

The $r'_{\alpha\beta}$ have the same properties as in the previous section and exactly as before the eigen-value λ_α of (65) is determined by (43) with

$$r' = \sum_{\alpha=1}^{m} r'_{\alpha\alpha}.$$

Bounds for r' are given by

$$\frac{\sum_{\alpha=1}^{m} r_{\alpha\alpha}}{(1+h_\beta^{(m)})_{\max}} \leqslant r' \leqslant \frac{\sum_{\alpha=1}^{m} r_{\alpha\alpha}}{(1+h_\beta^{(m)})_{\min}}.$$

6. Practical application

Usually in (43) $r'_{\alpha\alpha}$ will represent a small correction to the eigen-value, the terms

$$\pm r'_{\alpha\alpha} \frac{r'/t}{1-r'/t}$$

on the left and right can be neglected, and it is sufficiently accurate to evaluate $r'_{\alpha\alpha}$ for $\lambda = \lambda_\alpha^{(m)}$. The problem of estimating the proportional error in the approximation $\lambda_\alpha^{(m)}$ to λ_α is simply to find bounds for $r'_{\alpha\alpha}$. To be definite the first eigen-value λ_1 will be considered, and to simplify the notation the superscript (m) will be dropped except from $\lambda_1^{(m)}$, as the z_α, $x_{\alpha\gamma}$, $u_{\alpha\beta}$, $w_{\alpha\beta}$, h_α all refer to the curtailed system. It will be assumed that the eigen-vector $x_\alpha = x_{\alpha 1}$ has been calculated but not the $u_{\alpha\beta}$ and $w_{\alpha\beta}$.

To use (70) the value of $|z_1|$ is needed. The best result to be got from (61) without using the values of $u_{\alpha\beta}$ and $w_{\alpha\beta}$ follows from

$$\sum_{\alpha=1}^{m} |x_\alpha|^2 = \sum_{\beta=1}^{m} \frac{|w_{\beta 1}|^2}{1+h_\beta} |z_1|^2.$$

It is
$$(1+h_\beta)_{\min} \sum_{\alpha=1}^{m} |x_\alpha|^2 \leqslant |z_1|^2 \leqslant (1+h_\beta)_{\max} \sum_{\alpha=1}^{m} |x_\alpha|^2. \tag{71}$$

This can be used to rewrite (70) as

$$\frac{\sum_{\theta=m+1}^{\infty} |R_\theta|^2}{(1+h_\beta)_{\max} \sum_{\alpha=1}^{m} |x_\alpha|^2 (1+|\lambda|b_m)} \leqslant r'_{11} \leqslant \frac{\sum_{\theta=m+1}^{\infty} |R_\theta|^2}{(1+h_\beta)_{\min} \sum_{\alpha=1}^{m} |x_\alpha|^2 (1-|\lambda|b_m)}, \tag{72}$$

where
$$R_\theta = \sum_{\gamma=1}^{m} (\lambda k_{\gamma\theta} - h_{\gamma\theta}) \bar{x}_\gamma. \tag{73}$$

If $[h_{\alpha\beta}]$ is positive definite, the h_β are positive, and

$$(h_\beta)_{\max} \leqslant \sum_{\beta=1}^{m} h_\beta = \sum_{\beta=1}^{m} h_{\beta\beta}.$$

In this case (71) can be replaced by

$$\sum_{\alpha=1}^{m}|x_\alpha|^2 \leqslant |z_1|^2 \leqslant \Big(1+\sum_{\beta=1}^{m}h_{\beta\beta}\Big)\sum_{\alpha=1}^{m}|x_\alpha|^2, \tag{74}$$

and $(h_\beta)_{\max}$, $(h_\beta)_{\min}$ in (72) replaced by $\sum\limits_{\beta=1}^{m}h_{\beta\beta}$ and 0, respectively.

From (73)

$$\sum_{\theta=m+1}^{\infty}|R_\theta|^2 = \sum_{\alpha,\beta=1}^{m}\bar{x}_\alpha x_\beta \sum_{\theta=m+1}^{\infty}(\lambda^2 k_{\alpha\theta}\bar{k}_{\beta\theta}-\lambda k_{\alpha\theta}\bar{h}_{\beta\theta}-\lambda h_{\alpha\theta}\bar{k}_{\beta\theta}+h_{\alpha\theta}\bar{h}_{\beta\theta}). \tag{75}$$

It is convenient to consider separately the various terms on the right of (75). For the first term

$$\begin{aligned}\lambda^2\sum_{\alpha=1}^{m}|x_\alpha|^2\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2-\lambda^2\sum_{\alpha\neq\beta,\alpha,\beta=1}^{m}|x_\alpha||x_\beta|\Big\{\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}\Big\{\sum_{\theta=m+1}^{\infty}|k_{\beta\theta}|^2\Big\}^{\frac{1}{2}}\\ \leqslant \lambda^2\sum_{\alpha,\beta=1}^{m}\bar{x}_\alpha x_\beta\sum_{\theta=m+1}^{\infty}k_{\alpha\theta}\bar{k}_{\beta\theta} \\ \leqslant \lambda^2\sum_{\alpha=1}^{m}|x_\alpha|^2\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2+ \\ +\lambda^2\sum_{\alpha\neq\beta,\alpha,\beta=1}^{m}|x_\alpha||x_\beta|\Big\{\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}\Big\{\sum_{\theta=m+1}^{\infty}|k_{\beta\theta}|^2\Big\}^{\frac{1}{2}}.\end{aligned} \tag{76}$$

A similar result holds for the term

$$\sum_{\alpha,\beta=1}^{m}\bar{x}_\alpha x_\beta\sum_{\theta=m+1}^{\infty}h_{\alpha\theta}\bar{h}_{\beta\theta}.$$

Often the result (76) can be improved. Thus, if $k_{\alpha\theta}$ vanishes for $\alpha+\theta$ odd (even), then $\sum\limits_{\theta=m+1}^{\infty}k_{\alpha\theta}\bar{k}_{\beta\theta}$ vanishes for $\alpha+\beta$ odd (even) and the terms with $\alpha+\beta$ odd (even) can be omitted in the upper and lower bounds given by (76). For the term

$$-\lambda\sum_{\alpha,\beta=1}^{m}\bar{x}_\alpha x_\beta\sum_{\theta=m+1}^{\infty}k_{\alpha\theta}\bar{h}_{\beta\theta}$$

Schwarz's inequality gives simply

$$\begin{aligned}\Big|-\lambda\sum_{\alpha,\beta=1}^{m}\bar{x}_\alpha x_\beta\sum_{\theta=m+1}^{\infty}k_{\alpha\theta}\bar{h}_{\beta\theta}\Big| \\ \leqslant |\lambda|\sum_{\alpha,\beta=1}^{m}|x_\alpha||x_\beta|\Big\{\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}\Big\{\sum_{\theta=m+1}^{\infty}|h_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}.\end{aligned} \tag{77}$$

Again (77) can often be improved. If $k_{\alpha\theta}$ vanishes for $\alpha+\theta$ odd (even) and $h_{\beta\theta}$ vanishes for $\beta+\theta$ odd (even), the terms with $\alpha+\beta$ odd (even)

can be omitted. Similarly, if $k_{\alpha\theta}$ vanishes for $\alpha+\theta$ odd (even) and $h_{\beta\theta}$ for $\beta+\theta$ even (odd), the terms with $\alpha+\beta$ even (odd) can be omitted.

Use of (76) and (77) in (72) enables bounds to be found for r'_{11}. If there are no simplifying circumstances, the upper bound

$$\Big\{\sum_{\alpha=1}^{m}|x_\alpha|\Big[|\lambda|\Big\{\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}+\Big\{\sum_{\theta=m+1}^{\infty}|h_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}\Big]\Big\}^2 \tag{78}$$

is obtained for $$\sum_{\theta=m+1}^{\infty}|R_\theta|^2.$$

For very crude calculations Minkowski's inequality replaces this by

$$\sum_{\alpha=1}^{m}|x_\alpha|^2\Big[|\lambda|\Big\{\sum_{\alpha=1}^{m}\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}+\Big\{\sum_{\alpha=1}^{m}\sum_{\theta=m+1}^{\infty}|h_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}\Big]^2.$$

Similarly a lower bound is

$$2\lambda^2\sum_{\alpha=1}^{m}|x_\alpha|^2\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2+2\sum_{\alpha=1}^{m}|x_\alpha|^2\sum_{\theta=m+1}^{\infty}|h_{\alpha\theta}|^2-$$
$$-\Big\{\sum_{\alpha=1}^{m}|x_\alpha|\Big[|\lambda|\Big\{\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}+\Big\{\sum_{\theta=m+1}^{\infty}|h_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}\Big]\Big\}^2. \tag{79}$$

For calculations the values of

$$\Big\{\sum_{\theta=m+1}^{\infty}|k_{\alpha\theta}|^2\Big\}^{\frac{1}{2}},\qquad \Big\{\sum_{\theta=m+1}^{\infty}|h_{\alpha\theta}|^2\Big\}^{\frac{1}{2}}$$

and b_m are required. The first two will be simple to calculate either entirely numerically or by first obtaining an analytic expression for

$$\sum_{\theta=1}^{\infty}|k_{\alpha\theta}|^2.$$

Determining b_m may be more difficult, though the result

$$b_m \leqslant \Big\{\sum_{\alpha,\beta=m+1}^{\infty}|k_{\alpha\beta}|^2\Big\}^{\frac{1}{2}}+\frac{1}{|\lambda|}\Big\{\sum_{\alpha,\beta=m+1}^{\infty}|h_{\alpha\beta}|^2\Big\}^{\frac{1}{2}}$$

is useful. In general $|\lambda_1|b_m$ will be small and may be neglected in comparison with 1 if it cannot be estimated.

An alternative procedure to the use of (72) is to consider the determinantal equation analogous to (36),

$$\det\,(\delta_{\alpha\beta}-r_{\alpha\beta}+h_{\alpha\beta}-\lambda k_{\alpha\beta}) = 0. \tag{80}$$

If (80) is expanded and upper and lower limits are substituted for the $r_{\alpha\beta}$, making full use of any special properties the $r_{\alpha\beta}$ may have, more accurate estimates of the error in the approximation $\lambda_1^{(m)}$ will be obtained. This is because (72) suffers from the uncertainty in the value of $|z_1|^2$. However, expanding the determinantal equation (80)

involves more labour. In the case $[h_{\alpha\beta}] = 0$ the two procedures will give almost the same results.

Example. Leggett obtains the following system of equations*

$$(r^2+m^2)^2A_r - imQ\sum_{n=1}^{\infty} C_{rn}A_n + PR\frac{m^4}{(r^2+m^2)^2}A_r +$$

$$+PR\Big[L_r\sum_{n=1}^{\infty}\alpha_n A_n + M_r\sum_{n=1}^{\infty}(-1)^n\alpha_n A_n\Big] = 0 \quad (r = 1,2,...) \qquad (81)$$

in discussing the elastic stability of a long and slightly curved strip under shear. This system has a solution $(A_1, A_2,...)$ with not all the A_r zero for certain critical values of the shearing stress Q, the smallest of which corresponds to the buckling load. Leggett treats this system by the method of segments, comparing the approximations to the buckling load given by the second-, third-, and fourth-order determinants.

In (81), m is a parameter depending on the wave-length of the buckle, and PR depends on the curvature. To bring (81) into the notation of this paper, replace m, r, n by μ, α, β, and write $(\mu^2+\alpha^2)A_\alpha = x_\alpha$. Dividing (81) by $(\mu^2+\alpha^2)$ gives the form (4)

$$x_\alpha + \sum_{\beta=1}^{\infty} h_{\alpha\beta}x_\beta - \lambda\sum_{\beta=1}^{\infty} k_{\alpha\beta}x_\beta = 0 \quad (\alpha = 1,2,...), \qquad (82)$$

with
$$h_{\alpha\beta} = PRH^{(1)}_{\alpha\beta} + PRH^{(2)}_{\alpha\beta}, \qquad (83)$$

$$k_{\alpha\beta}\begin{cases} = i\,\dfrac{2\alpha\beta[1-(-1)^{\alpha+\beta}]}{(\alpha^2-\beta^2)(\mu^2+\alpha^2)(\mu^2+\beta^2)} & (\alpha \neq \beta), \\ = 0 & (\alpha = \beta), \end{cases}$$

$$\lambda = \frac{\mu Q}{\pi},$$

$$H^{(1)}_{\alpha\beta} = \frac{\mu^4}{(\mu^2+\alpha^2)^4}\delta_{\alpha\beta},$$

$$H^{(2)}_{\alpha\beta} = [1+(-1)^{\alpha+\beta}][a+b(-1)^\alpha]H_\alpha H_\beta,$$

$$H_\alpha = \frac{\alpha(\alpha^2+2\cdot 25\mu^2)}{(\mu^2+\alpha^2)^3},$$

and a, b complicated functions of μ.

The work necessary in estimating the error of the second approximation (from the third-order determinant) will be outlined. This error

* (5), 17 (52).

can then be compared with the difference between the second and third approximations.

It will be seen that $[h_{\alpha\beta}]$ and $[k_{\alpha\beta}]$ are Hermitian, and that $h_{\alpha\beta}$ vanishes for $\alpha+\beta$ odd, $k_{\alpha\beta}$ for $\alpha+\beta$ even. By (76) and (77), using the simplifications due to the special properties of $h_{\alpha\beta}$ and $k_{\alpha\beta}$, an upper bound for

$$\sum_{\theta=m+1}^{\infty} |R_\theta|^2 \quad (m = 3),$$

is the following expression with the upper signs throughout and a lower bound is the same expression with the lower signs throughout.

$$\lambda^2\Big[|x_1|^2 \sum_{\theta=4}^{\infty} |k_{1\theta}|^2 + |x_2|^2 \sum_{\theta=4}^{\infty} |k_{2\theta}|^2 + |x_3|^2 \sum_{\theta=4}^{\infty} |k_{3\theta}|^2\Big] +$$

$$+|x_1|^2 \sum_{\theta=4}^{\infty} |h_{1\theta}|^2 + |x_2|^2 \sum_{\theta=4}^{\infty} |h_{2\theta}|^2 + |x_3|^2 \sum_{\theta=4}^{\infty} |h_{3\theta}|^2 \pm$$

$$\pm 2\lambda^2 |x_1||x_3| \Big\{\sum_{\theta=4}^{\infty} |k_{1\theta}|^2\Big\}^{\frac{1}{2}} \Big\{\sum_{\theta=4}^{\infty} |k_{3\theta}|^2\Big\}^{\frac{1}{2}} \pm 2|x_1||x_3| \Big\{\sum_{\theta=4}^{\infty} |h_{1\theta}|^2\Big\}^{\frac{1}{2}} \Big\{\sum_{\theta=4}^{\infty} |h_{3\theta}|^2\Big\}^{\frac{1}{2}} \pm$$

$$\pm 2|\lambda||x_1||x_2| \Big[\Big\{\sum_{\theta=4}^{\infty} |k_{1\theta}|^2\Big\}^{\frac{1}{2}} \Big\{\sum_{\theta=4}^{\infty} |h_{2\theta}|^2\Big\}^{\frac{1}{2}} + \Big\{\sum_{\theta=4}^{\infty} |k_{2\theta}|^2\Big\}^{\frac{1}{2}} \Big\{\sum_{\theta=4}^{\infty} |h_{1\theta}|^2\Big\}^{\frac{1}{2}}\Big] \pm$$

$$\pm 2|\lambda||x_2||x_3| \Big[\Big\{\sum_{\theta=4}^{\infty} |k_{2\theta}|^2\Big\}^{\frac{1}{2}} \Big\{\sum_{\theta=4}^{\infty} |h_{3\theta}|^2\Big\}^{\frac{1}{2}} + \Big\{\sum_{\theta=4}^{\infty} |k_{3\theta}|^2\Big\}^{\frac{1}{2}} \Big\{\sum_{\theta=4}^{\infty} |h_{2\theta}|^2\Big\}^{\frac{1}{2}}\Big].$$

Of course, if the lower bound given by this expression is negative, it should be replaced by zero.

The calculation of

$$\sum_{\theta=m+1}^{\infty} |k_{\alpha\theta}|^2, \qquad \sum_{\theta=m+1}^{\infty} |h_{\alpha\theta}|^2$$

is simple. Write $\sum\limits_{\theta=m+1}^{\infty} |k_{\alpha\theta}|^2$ as $\sum\limits_{\theta=1}^{\infty} |k_{\alpha\theta}|^2 - \sum\limits_{\theta=1}^{m} |k_{\alpha\theta}|^2$. The quantities $|k_{\alpha\theta}|$ $(\alpha, \theta = 1,..., m)$ are known from the work on the curtailed system. For $\sum\limits_{\theta=1}^{\infty} |k_{\alpha\theta}|^2$ it is easy to obtain the result

$$\sum_{\theta=1}^{\infty} |k_{\alpha\theta}|^2 = \begin{cases} (\frac{1}{2}\pi)^2 \dfrac{4\alpha^2}{(\mu^2+\alpha^2)^4}\Big[1 - \dfrac{1}{\sinh^2 \frac{1}{2}\pi\mu} - \dfrac{\coth \frac{1}{2}\pi\mu}{\frac{1}{2}\pi\mu} \dfrac{3\mu^2-\alpha^2}{\mu^2+\alpha^2}\Big] & (\alpha \text{ odd}), \\ (\frac{1}{2}\pi)^2 \dfrac{4\alpha^2}{(\mu^2+\alpha^2)^4}\Big[1 + \dfrac{1}{\cosh^2 \frac{1}{2}\pi\mu} - \dfrac{\tanh \frac{1}{2}\pi\mu}{\frac{1}{2}\pi\mu} \dfrac{3\mu^2-\alpha^2}{\mu^2+\alpha^2}\Big] & (\alpha \text{ even}) \end{cases}$$

by standard methods of summing series. For $\sum_{\theta=m+1}^{\infty} |h_{\alpha\theta}|^2$

$$\sum_{\theta=m+1}^{\infty} |h_{\alpha\theta}|^2 = 4(PR)^2[a+b(-1)^\alpha]^2(H_\alpha)^2 \sum_{\theta=m+1}^{\infty} \left(\frac{1+(-1)^{\alpha+\theta}}{2} H_\theta\right)^2$$

$$= \begin{cases} 4(PR)^2[a-b]^2(H_\alpha)^2 \sum_{\text{odd}\,\theta>m}^{\infty} (H_\theta)^2 & (\alpha \text{ odd}), \\ 4(PR)^2[a+b]^2(H_\alpha)^2 \sum_{\text{even}\,\theta>m}^{\infty} (H_\theta)^2 & (\alpha \text{ even}). \end{cases}$$

It is then best to calculate

$$\sum_{\text{odd}\,\theta>m}^{\infty} (H_\theta)^2, \qquad \sum_{\text{even}\,\theta>m}^{\infty} (H_\theta)^2$$

numerically.

Further $(h_\beta)_{\max}$, $(h_\beta)_{\min}$ $(\beta = 1, 2, 3)$ are required. These are the greatest and least of μ_1, h_{22}, μ_3, where μ_1, μ_3 are the roots of

$$\begin{vmatrix} -\mu+h_{11} & h_{13} \\ h_{31} & -\mu+h_{33} \end{vmatrix} = 0.$$

Of course, it is easy to see that $[h_{\alpha\beta}]$ is positive definite, so that the cruder estimate (74) can be used. For

$$\sum_{\alpha,\beta=1}^{m} h_{\alpha\beta} x_\alpha x_\beta = PR \sum_{\alpha=1}^{m} H_{\alpha\alpha}^{(1)}(x_\alpha)^2 + 2PR(a-b)\Big[\sum_{\text{odd}\,\alpha=1}^{m} H_\alpha x_\alpha\Big]^2 + $$
$$+2PR(a+b)\Big[\sum_{\text{even}\,\alpha<2}^{m} H_\alpha x_\alpha\Big]^2$$

and each of the expressions on the right is positive.

Finally the value of $|\lambda| b_m$ is required. Let $[a_{\alpha\beta}]_m$ be the matrix $[a_{\alpha\beta}]$ with the first m rows and first m columns removed, and let $b[a_{\alpha\beta}]_m$ be the Hilbert bound of $[a_{\alpha\beta}]_m$. The more important properties of bounds are given in (**1, 2, 3**). We have

$$\begin{aligned} |\lambda| b_m &= b[h_{\alpha\beta} - \lambda k_{\alpha\beta}]_m \\ &\leqslant b[h_{\alpha\beta}]_m + |\lambda| b[k_{\alpha\beta}]_m \\ &\leqslant PRb[H_{\alpha\beta}^{(1)}]_m + PRb[H_{\alpha\beta}^{(2)}]_m + |\lambda| b[k_{\alpha\beta}]_m. \end{aligned}$$

Now $b[a_{\alpha\beta}]$ is the maximum value of

$$\Big|\sum_{\alpha,\beta=1}^{\infty} a_{\alpha\beta} x_\alpha y_\beta\Big|,$$

where $$\sum_{\alpha=1}^{\infty} |x_\alpha|^2 = 1, \qquad \sum_{\beta=1}^{\infty} |y_\beta|^2 = 1.$$

If $a_{\alpha\beta}=a_{\beta\alpha}$, and $a_{\alpha\beta}$ is real, the maximum is attained for $x_\alpha=y_\alpha$ and y_α real. For $[H^{(1)}_{\alpha\beta}]_m$, $b[H^{(1)}_{\alpha\beta}]_m$ is the maximum value of

$$\left|\sum_{\alpha=m+1}^{\infty}\frac{\mu^4(x_\alpha)^2}{(\mu^2+\alpha^2)^4}\right|$$

under the condition $$\sum_{\alpha=m+1}^{\infty}(x_\alpha)^2=1.$$

Obviously $$b[H^{(1)}_{\alpha\beta}]_m=\frac{\mu^4}{\{\mu^2+(m+1)^2\}^4}.$$

For $[H^{(2)}_{\alpha\beta}]_m$, we have

$$\begin{aligned}
b[H^{(2)}_{\alpha\beta}]_m &= \max\Big|\sum_{\text{even}\,\alpha,\beta>m}^{\infty}2(a+b)(H_\alpha x_\alpha)(H_\beta x_\beta)+\\
&\qquad\qquad+\sum_{\text{odd}\,\alpha,\beta>m}^{\infty}2(a-b)(H_\alpha x_\alpha)(H_\beta x_\beta)\Big|\\
&\leqslant \max\Big\{2|a+b|\Big|\sum_{\text{even}\,\alpha,\beta>m}^{\infty}(H_\alpha x_\alpha)(H_\beta x_\beta)\Big|+\\
&\qquad\qquad+2|a-b|\Big|\sum_{\text{odd}\,\alpha,\beta>m}^{\infty}(H_\alpha x_\alpha)(H_\beta x_\beta)\Big|\Big\}\\
&\leqslant \max\Big\{2|a+b|\sum_{\text{even}\,\alpha>m}^{\infty}(H_\alpha)^2\sum_{\text{even}\,\alpha>m}^{\infty}(x_\alpha)^2+\\
&\qquad\qquad+2|a-b|\sum_{\text{odd}\,\alpha>m}^{\infty}(H_\alpha)^2\sum_{\text{odd}\,\alpha>m}^{\infty}(x_\alpha)^2\Big\}\\
&= \max\Big\{2|a+b|\sum_{\text{even}\,\alpha>m}^{\infty}(H_\alpha)^2,\ 2|a-b|\sum_{\text{odd}\,\alpha>m}^{\infty}(H_\alpha)^2\Big\}.
\end{aligned}$$

For $[k_{\alpha\beta}]_m$, more difficulty is experienced, and appeal is made to a series-inversion formula due to Titchmarsh (**6**). This states that, if

$$W_\alpha=\frac{1}{\pi}\sum_{\beta=-\infty}^{\infty}\frac{U_\beta}{\alpha+\beta+\frac{1}{2}},$$

then $$U_\beta=\frac{1}{\pi}\sum_{\alpha=-\infty}^{\infty}\frac{W_\alpha}{\alpha+\beta+\frac{1}{2}}$$

and $$\sum_{\alpha=-\infty}^{\infty}W_\alpha^2=\sum_{\beta=-\infty}^{\infty}U_\beta^2.$$

Some preliminary transformations will first be made. We have

$$\sum_{\alpha,\beta=m+1}^{\infty}k_{\alpha\beta}x_\alpha y_\beta=i\sum_{\alpha,\beta=m+1}^{\infty}\frac{2\alpha\beta[(-1)^\alpha-(-1)^\beta]}{(\alpha^2-\beta^2)(\mu^2+\alpha^2)(\mu^2+\beta^2)}(-1)^\alpha x_\alpha y_\beta.$$

Writing $(-1)^\alpha x_\alpha = z_\alpha$, we get

$$\left|\sum_{\alpha,\beta=m+1}^{\infty} k_{\alpha\beta}x_\alpha y_\beta\right| = \left|\sum_{\alpha,\beta=m+1}^{\infty} \frac{2\alpha\beta[(-1)^\alpha-(-1)^\beta]}{(\alpha^2-\beta^2)(\mu^2+\alpha^2)(\mu^2+\beta^2)} z_\alpha y_\beta\right|.$$

The expression on the right is a bilinear form with symmetric real coefficients, and therefore attains its bound for $z_\alpha = y_\alpha$ and y_α real. Thus

$$b[k_{\alpha\beta}]_m = \max\left|\sum_{\alpha,\beta=m+1}^{\infty} \frac{2\alpha\beta[(-1)^\alpha-(-1)^\beta]}{(\alpha^2-\beta^2)(\mu^2+\alpha^2)(\mu^2+\beta^2)} y_\alpha y_\beta\right|,$$

where
$$\sum_{\alpha=m+1}^{\infty}(y_\alpha)^2 = 1.$$

Since $k_{\alpha\beta}$ vanishes for $\alpha+\beta$ even, this can be written as

$$b[k_{\alpha\beta}]_m = 4\max\left|\sum_{\substack{\text{even}\,\alpha>m\\ \text{odd}\,\beta>m}}^{\infty} \frac{2\alpha\beta}{(\alpha^2-\beta^2)(\mu^2+\alpha^2)(\mu^2+\beta^2)} y_\alpha y_\beta\right|.$$

Now

$$\sum_{\substack{\text{even}\,\alpha>m\\ \text{odd}\,\beta>m}}^{\infty} \frac{2\alpha\beta y_\alpha y_\beta}{(\alpha^2-\beta^2)(\mu^2+\alpha^2)(\mu^2+\beta^2)}$$
$$= -\sum_{\substack{\text{even}\,\alpha>m\\ \text{odd}\,\beta>m}}^{\infty} \frac{\alpha}{(\mu^2+\alpha^2)(\mu^2+\beta^2)}\left(\frac{1}{\alpha+\beta}+\frac{1}{-\alpha+\beta}\right) y_\alpha y_\beta.$$

Let $\alpha = 2\gamma$, $\beta = 2\epsilon+1$, $u_\gamma = y_\alpha$, $v_\epsilon = y_\beta$,

$$\sum_{\substack{\text{even}\,\alpha>m\\ \text{odd}\,\beta>m}}^{\infty} \frac{2\alpha\beta y_\alpha y_\beta}{(\alpha^2-\beta^2)(\mu^2+\alpha^2)(\mu^2+\beta^2)}$$
$$= -\sum_{\substack{\gamma>\frac{1}{2}m\\ \epsilon>\frac{1}{2}m}}^{\infty} \frac{\gamma u_\gamma v_\epsilon}{(\mu^2+4\gamma^2)\{\mu^2+(2\epsilon+1)^2\}}\left(\frac{1}{\gamma+\epsilon+\frac{1}{2}}+\frac{1}{-\gamma+\epsilon+\frac{1}{2}}\right).$$

Let
$$\frac{\gamma u_\gamma}{\mu^2+4\gamma^2} = U_\gamma, \qquad \frac{v_\epsilon}{\mu^2+(2\epsilon+1)^2} = V_\epsilon.$$

Then
$$b[k_{\alpha\beta}]_m = 4\max\left|\sum_{\substack{\gamma>\frac{1}{2}m\\ \epsilon>\frac{1}{2}m}}^{\infty} U_\gamma V_\epsilon\left(\frac{1}{\gamma+\epsilon+\frac{1}{2}}+\frac{1}{-\gamma+\epsilon+\frac{1}{2}}\right)\right|.$$

Defining $U_\gamma = 0$ ($|\gamma| \leqslant \frac{1}{2}m$), $U_{-\gamma} = U_\gamma$, we have

$$b[k_{\alpha\beta}]_m = 4\max\left|\sum_{\epsilon>\frac{1}{2}m}^{\infty}\sum_{\gamma=-\infty}^{\infty}\frac{U_\gamma V_\epsilon}{\gamma+\epsilon+\frac{1}{2}}\right|$$

$$= 4\pi\max\left|\sum_{\epsilon>\frac{1}{2}m}^{\infty} V_\epsilon W_\epsilon\right|$$

$$\leqslant 4\pi\max\left\{\sum_{\epsilon\geqslant\frac{1}{2}m}^{\infty} V_\epsilon^2\right\}^{\frac{1}{2}}\left\{\sum_{\epsilon\geqslant\frac{1}{2}m}^{\infty} W_\epsilon^2\right\}^{\frac{1}{2}}$$

$$\leqslant 4\pi\max\left\{\sum_{\epsilon>\frac{1}{2}m}^{\infty} V_\epsilon^2\right\}^{\frac{1}{2}}\left\{\sum_{\epsilon=0}^{\infty} W_\epsilon^2\right\}^{\frac{1}{2}},$$

where
$$W_\epsilon = \sum_{\gamma=-\infty}^{\infty}\frac{U_\gamma}{\gamma+\epsilon+\frac{1}{2}}$$

and
$$\sum_{\epsilon=0}^{\infty} W_\epsilon^2 = \frac{1}{2}\sum_{\epsilon=-\infty}^{\infty} W_\epsilon^2 = \frac{1}{2}\sum_{\epsilon=-\infty}^{\infty} U_\gamma^2 = \sum_{\gamma>\frac{1}{2}m}^{\infty} U_\gamma^2.$$

The condition
$$\sum_{\alpha=m+1}^{\infty} y_\alpha^2 = \sum_{\gamma>\frac{1}{2}m}^{\infty} u_\gamma^2 + \sum_{\epsilon>\frac{1}{2}m}^{\infty} v_\epsilon^2 = 1$$

gives
$$\sum_{\gamma>\frac{1}{2}m}^{\infty}\left[\frac{(\mu^2+4\gamma^2)U_\gamma}{\gamma}\right]^2 + \sum_{\epsilon>\frac{1}{2}m}^{\infty}[\mu^2+(2\epsilon+1)^2]^2 V_\epsilon^2 = 1,$$

which will be written as $\quad U+V = 1.$

Obviously
$$b[k_{\alpha\beta}]_m \leqslant \left[\frac{4\pi\gamma}{(\mu^2+4\gamma^2)\{\mu^2+(2\epsilon+1)^2\}}\right]_{\max} U^{\frac{1}{2}}V^{\frac{1}{2}}$$

$$\leqslant \pi\left\{\frac{\gamma}{\mu^2+4\gamma^2}\right\}_{\substack{\max\\ \gamma>\frac{1}{2}m}}\left\{\frac{1}{\mu^2+(2\epsilon+1)^2}\right\}_{\substack{\max\\ \epsilon\geqslant\frac{1}{2}m}}.$$

It is thus simple to evaluate $|\lambda|b_m$. Substituting the lower bound for $\sum_{\theta=m+1}^{\infty}|R_\theta|^2$ on the left of (72), and the upper bound for $\sum_{\theta=m+1}^{\infty}|R_\theta|^2$ on the right gives bounds for r'_{11}.

Table 1 gives upper bounds (u.b.) and lower bounds (l.b.) for r'_{11} and upper bounds for r' for various values of PR and μ; also upper and lower bounds for the fractional error. The upper bound for r' is calculated from

$$r' \leqslant \frac{\sum_{\alpha=1}^{m} r_{\alpha\alpha}}{(1+h_\beta)_{\min}} \leqslant \frac{\sum_{\alpha=1}^{m}\sum_{\beta=m+1}^{\infty}[|h_{\alpha\beta}|^2+\lambda^2|k_{\alpha\beta}|^2]}{(1+h_\beta)_{\min}(1-|\lambda|b_m)}.$$

The result (44) is applied to obtain the extreme possible values of

the error. The eigen-values $\lambda_1, \lambda_2, \lambda_3$ for the third-order determinant are $\lambda_1, -\lambda_1, \infty$ and therefore

$$t = \min|1-\lambda/\lambda_\alpha| = 1 \quad (\alpha \neq 1,\ 0 < \lambda/\lambda_1 \leqslant 1).$$

TABLE 1

$\sqrt{(PR)}$	μ	*l.b. for* r'_{11}	*u.b. for* r'_{11}	*u.b. for* r'	*l.b. for error*	*u.b. for error*
0	0·794	0·0036	0·0117	0·0300	0·0036	0·0120
1	0·911	0·0023	0·0170	0·0433	0·0023	0·0178
2	1·072	0·0009	0·0295	0·0718	0·0009	0·0318
3	1·162	0·0007	0·0483	0·1065	0·0007	0·0541
4	1·245	0·0013	0·0728	0·1481	0·0013	0·0854
5·5	1·323	0·0024	0·1213	0·2215	0·0024	0·1558
7	1·323	0·0030	0·1801	0·3068	0·0030	0·2599

A direct arithmetical computation of $\sum_{\theta=m+1}^{\infty} R_\theta^2$ determines closer bounds for r'_{11} and thus closer bounds for the error. The results are given in Table 2, the same upper bounds for r' as in Table 1 being used to calculate the bounds for the error. The fractional difference between the second and third approximations $(1-\lambda_1^{(4)}/\lambda_1^{(3)})$ is also included.

TABLE 2

$\sqrt{(PR)}$	μ	*l.b. for* r'_{11}	*u.b. for* r'_{11}	*l.b. for error*	*u.b. for error*	$1-\lambda_1^{(3)}/\lambda_1^{(2)}$
0	0·794	0·0037	0·0040	0·0037	0·0041	0·0023
1	0·911	0·0027	0·0037	0·0027	0·0039	0·0026
2	1·072	0·0022	0·0044	0·0022	0·0047	—
3	1·162	0·0031	0·0088	0·0031	0·0099	—
4	1·245	0·0051	0·0194	0·0051	0·0228	0·0106
5·5	1·323	0·0081	0·0464	0·0081	0·0595	—
7	1·323	0·0099	0·0868	0·0099	0·1253	0·0466

Closer bounds for the error can be obtained by changing the variables in (82) to

$$y_\alpha = x_\alpha(1+PRH_{\alpha\alpha}^{(1)})^{\frac{1}{2}}$$

and dividing the αth equation by $(1+PRH_{\alpha\alpha}^{(1)})^{\frac{1}{2}}$ or, alternatively, to the variables

$$y_\alpha = x_\alpha(1+PRH_{\alpha\alpha}^{(1)})^{\frac{1}{2}} \quad (\alpha \leqslant m),$$
$$y_\alpha = x_\alpha \quad (\alpha > m)$$

and dividing each of the first m equations by

$$(1+PRH_{\alpha\alpha}^{(1)})^{\frac{1}{2}} \quad (\alpha = 1,\ldots, m).$$

The second procedure has the advantage that most of the manipulation and the arithmetic used in obtaining Tables 1 and 2 can be taken over unchanged. Table 3 gives the results from this method. Again $\sum R_\theta^2$ was computed directly. An exact value is, of course, preferable to upper and lower bounds and is perhaps simpler to calculate.

TABLE 3

$\sqrt{(PR)}$	μ	*l.b. for* r'_{11}	*u.b. for* r'_{11}	*u.b. for* r'	*l.b. for error*	*u.b. for error*
0	0·794	0·0037	0·0040	0·0300	0·0037	0·0041
1	0·911	0·0028	0·0036	0·0423	0·0028	0·0039
2	1·072	0·0026	0·0041	0·0670	0·0026	0·0044
3	1·162	0·0042	0·0079	0·0952	0·0042	0·0087
4	1·245	0·0083	0·0170	0·1284	0·0083	0·0195
5·5	1·323	0·0174	0·0396	0·1605	0·0174	0·0486
7	1·323	0·0285	0·0717	0·2460	0·0285	0·0950

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SOME FOURIER TRANSFORMS IN PRIME-NUMBER THEORY

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1. Introduction

It has been shown that, if the Riemann hypothesis is true, then the relationship between the distribution of the non-trivial zeros of the Riemann zeta-function and that of the logarithms of the powers of prime numbers can be expressed in various ways closely connected with Fourier and Hankel transformations.†

In the present paper I discuss two pairs of Fourier cosine-transforms which illustrate one aspect of this relationship. I have previously discussed a related pair of Hankel transforms,‡ but it is of some interest to construct a result which only involves the simpler Fourier cosine kernel. In each of the present pairs of transforms one function has simple discontinuities of magnitude $(2\pi)^{\frac{1}{2}}/x$ when the argument x passes through a zero of $\zeta(\frac{1}{2}+ix)$, while the transform has simple discontinuities of magnitude $1/mp^{\frac{1}{2}m}$ when the argument passes through a value of $\log p^m$, where p^m is a positive integral power of a prime p.

I also derive an alternative proof of an infinite-series formula for $N(T)$, the number of zeros of $\zeta(s)$ in $0 < I(s) < T$.

All the results described above require the assumption of the Riemann hypothesis. Some simpler analogous results requiring no unproved hypothesis are given in the last section.

2. First pair of transforms

Suppose that the Riemann hypothesis is true, and let $\frac{1}{2}\pm i\gamma_n$ $(n = 1, 2, 3,...;\ 0 < \gamma_n \leqslant \gamma_{n+1})$ run through the non-trivial zeros of $\zeta(s)$.§ Then

$$\sum_{\gamma_n<x}\frac{1}{\gamma_n} = \int_1^x \frac{dN(t)}{t} = \frac{N(x)}{x}+\int_1^x N(t)\frac{dt}{t^2} \tag{2.1}$$

† A. Wintner, *Duke J. of Math.* 10 (1943), 99–105 (99), and A. P. Guinand, *Proc. Lond. Math. Soc.* (to appear shortly), referred to in the sequel as (A).

‡ (A), Theorem 1.

§ If $\frac{1}{2}+i\gamma_n$ is a multiple zero of $\zeta(s)$ of order $r+1$ then we put

$$\gamma_n = \gamma_{n+1} = \dots = \gamma_{n+r}.$$

since $$\gamma_1 = 14{\cdot}13 > 1.$$

Now† $$N(x) = \frac{x}{2\pi}\log\frac{x}{2\pi} - \frac{x}{2\pi} + \frac{7}{8} + R(x),$$

where $$R(x) = O(\log x).$$

Hence (2.1) becomes

$$\frac{1}{2\pi}\log\frac{x}{2\pi} - \frac{1}{2\pi} + \frac{R(x)}{x} + \int_1^x \left(\log\frac{t}{2\pi} - 1\right)\frac{dt}{2\pi t} + \int_1^x R(t)\frac{dt}{t^2} + \frac{7}{8x} + \frac{7}{8}\int_1^x \frac{dt}{t^2}$$

$$= \frac{1}{4\pi}\log^2 x - \frac{1}{2\pi}\log 2\pi \log x - \frac{1}{2\pi}(1+\log 2\pi) +$$

$$+ \int_1^\infty R(t)\frac{dt}{t^2} + \frac{7}{8} - \int_x^\infty R(t)\frac{dt}{t^2}$$

$$= \frac{1}{4\pi}\log^2 x - \frac{1}{2\pi}\log 2\pi \log x + k + O\left(\frac{\log x}{x}\right), \quad (2.2)$$

where

$$k = \int_1^\infty R(t)\frac{dt}{t^2} - \frac{1}{2\pi}(1+\log 2\pi) + \frac{7}{8}$$

$$= \lim_{x\to\infty}\left\{\sum_{\gamma_n < x}\frac{1}{\gamma_n} - \frac{1}{4\pi}\log^2 x + \frac{1}{2\pi}\log 2\pi \log x\right\}.$$

Now put

$$F(x) = (2\pi)^{\frac{1}{2}}\left\{\sum_{\gamma_n \leqslant x}{}' \frac{1}{\gamma_n} - \frac{1}{4\pi}\log^2 x + \frac{1}{2\pi}\log 2\pi \log x - k\right\},$$

where the dash indicates that the terms $\gamma_n = x$, if they occur, are to be halved. Then

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\gamma_N} F(t)\cos xt\, dt$$

$$= 2\sum_{n=1}^{N-1}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \ldots + \frac{1}{\gamma_n}\right)\int_{\gamma_n}^{\gamma_{n+1}} \cos xt\, dt -$$

$$- \frac{1}{2\pi}\int_0^{\gamma_N} \log^2 t \cos xt\, dt + \frac{1}{\pi}\log 2\pi \int_0^{\gamma_N} \log t \cos xt\, dt - 2k\int_0^{\gamma_N} \cos xt\, dt$$

† E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 4.

$$= \frac{2}{x}\sum_{n=1}^{N-1}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\ldots+\frac{1}{\gamma_n}\right)(\sin x\gamma_{n+1}-\sin x\gamma_n)-$$

$$-\frac{1}{2\pi x}\log^2\gamma_N\sin x\gamma_N+\frac{1}{\pi x}\int\limits_0^{\gamma_N}\log t\sin xt\frac{dt}{t}+$$

$$+\frac{1}{\pi x}\log 2\pi\log\gamma_N\sin x\gamma_N-\frac{1}{\pi x}\log 2\pi\int\limits_0^{\gamma_N}\sin xt\frac{dt}{t}-\frac{2k}{x}\sin x\gamma_N$$

$$= -\frac{2}{x}\sum_{n=1}^{N}\frac{\sin x\gamma_n}{\gamma_n}+$$

$$+\frac{2}{x}\sin x\gamma_N\left\{\sum_{n=1}^{N}\frac{1}{\gamma_n}-\frac{1}{4\pi}\log^2\gamma_N+\frac{1}{2\pi}\log 2\pi\log\gamma_N-k\right\}+$$

$$+\frac{1}{\pi x}\int\limits_0^{\infty}\log t\sin xt\frac{dt}{t}-\frac{1}{\pi x}\log 2\pi\int\limits_0^{\infty}\sin xt\frac{dt}{t}+O\left(\frac{\log\gamma_N}{\gamma_N}\right). \tag{2.3}$$

Now, by (2.2), the expression in the brackets $\{\quad\}$ is $O\left(\frac{\log\gamma_N}{\gamma_N}\right)$. Also

$$\int\limits_0^{\infty}\log t\sin xt\frac{dt}{t} = \int\limits_0^{\infty}\log u\sin u\frac{du}{u}-\log x\int\limits_0^{\infty}\sin u\frac{du}{u}$$

$$= -\tfrac{1}{2}\pi(C+\log x),$$

where C is Euler's constant. Hence (2.3) becomes

$$-\frac{2}{x}\sum_{n=1}^{N}\frac{\sin x\gamma_n}{\gamma_n}-\frac{1}{2x}(C+\log 2\pi x)+O\left(\frac{\log\gamma_N}{\gamma_N}\right). \tag{2.4}$$

Now it is known that†

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p^m\leqslant y}{}'\frac{\log p}{p^{ms}}-\frac{y^{1-s}}{1-s}-\sum_{q=1}^{\infty}\frac{y^{-2q-s}}{2q+s}+\sum_{\rho}\frac{y^{\rho-s}}{\rho-s},$$

where ρ runs through the non-trivial zeros of $\zeta(s)$; $s\neq 1, -2q, \rho$; and $y>1$. If we put $s=\frac{1}{2}$, $\rho=\frac{1}{2}\pm i\gamma_n$, $y=e^x$, then it follows, after some manipulation,‡ that the series

$$2\sum_{n=1}^{\infty}\frac{\sin x\gamma_n}{\gamma_n}$$

† E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 81.

‡ See (A) for details.

converges to the sum

$$-\sum_{m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}}+4\sinh\tfrac{1}{2}x+\tfrac{1}{2}\log\coth\tfrac{1}{4}x+$$
$$+\arctan e^{-\frac{1}{2}x}-\tfrac{1}{2}C-\tfrac{1}{4}\pi-\tfrac{1}{2}\log 8\pi. \qquad (2.5)$$

Substituting (2.5) in (2.4) and making N tend to infinity, we have, after rearrangement,

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_{0}^{\to\infty} F(t)\cos xt\,dt$$
$$=\frac{1}{x}\left\{\sum_{m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}}-4\sinh\tfrac{1}{2}x+\tfrac{1}{2}\log\left(\frac{\tanh\frac{1}{4}x}{\frac{1}{4}x}\right)+(\tfrac{1}{4}\pi-\arctan e^{-\frac{1}{2}x})\right\}$$
$$=G(x), \text{ say.}$$

That is, $G(x)$ is the Fourier cosine-transform of $F(x)$, and the integral converges in the ordinary sense.

Further, it follows from (2.2) that $F(x)$ belongs to $L^2(0,\infty)$. Hence $G(x)$ also belongs to $L^2(0,\infty)$, and, since $F(x)$ is of bounded variation in any finite interval excluding the origin, it follows by a theorem of Titchmarsh† that

$$F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_{0}^{\to\infty} G(t)\cos xt\,dt.$$

Thus we have

THEOREM 1. *If the Riemann hypothesis is true, and*

$$F(x)=(2\pi)^{\frac{1}{2}}\left\{\sum_{\gamma_n\leqslant x}{}' \frac{1}{\gamma_n}-\frac{1}{4\pi}\log^2 x+\frac{1}{2\pi}\log 2\pi\log x-k\right\},$$

where k is chosen so that $\lim_{x\to\infty} F(x)=0$, and

$$G(x)=\frac{1}{x}\left\{\sum_{m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}}-4\sinh\tfrac{1}{2}x+\tfrac{1}{2}\log\left(\frac{\tanh\frac{1}{4}x}{\frac{1}{4}x}\right)+\tfrac{1}{4}\pi-\arctan e^{-\frac{1}{2}x}\right\},$$

then, for $x>0$,

$$F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_{0}^{\to\infty} G(t)\cos xt\,dt,$$

and

$$G(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_{0}^{\to\infty} F(t)\cos xt\,dt.$$

† E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), Theorem 58.

3. Second pair of transforms

We use the following lemma:†

LEMMA. *If $f(x)$ and $g(x)$ belong to $L^2(0,\infty)$ and are a pair of Fourier cosine-transforms, then the functions*

$$f(x)-\int_x^\infty f(t)\frac{dt}{t}, \qquad g(x)-\frac{1}{x}\int_0^x g(t)\,dt$$

are also a pair of Fourier cosine-transforms belonging to $L^2(0,\infty)$.

Now suppose that $\{\alpha_n\}$ is an increasing sequence of positive numbers tending to infinity, that $\{a_n\}$ is another sequence not necessarily increasing or positive, that $A(x)$ is everywhere differentiable, and that

$$f(x)=\frac{1}{x}\Big\{\sum_{\alpha_n\leqslant x}{}' a_n-A(x)\Big\} \tag{3.1}$$

belongs to $L^2(0,\infty)$. If we choose M so that $\alpha_{M-1}<x\leqslant\alpha_M$, and put $N>M$, $\alpha_N=T$, then

$$\int_x^T f(t)\frac{dt}{t}=(a_1+a_2+...+a_{M-1})\int_x^{\alpha_M}\frac{dt}{t^2}+$$

$$+\sum_{n=M}^{N-1}(a_1+a_2+...+a_n)\int_{\alpha_n}^{\alpha_{n+1}}\frac{dt}{t^2}-\int_x^T A(t)\frac{dt}{t^2}$$

$$=\Big(\frac{1}{x}-\frac{1}{\alpha_M}\Big)\sum_{n=1}^{M-1}a_n+\sum_{n=M}^{N-1}(a_1+a_2+...+a_n)\Big(\frac{1}{\alpha_n}-\frac{1}{\alpha_{n+1}}\Big)+$$

$$+\Big[\frac{A(t)}{t}\Big]_x^T-\int_x^T A'(t)\frac{dt}{t}$$

$$=\sum_{n=M}^{N}\frac{a_n}{\alpha_n}-\frac{1}{\alpha_N}\sum_{n=1}^{N}a_n+\frac{1}{x}\sum_{n=1}^{M-1}a_n+$$

$$+\frac{A(T)}{T}-\frac{A(x)}{x}-\int_x^T A'(t)\frac{dt}{t}$$

† This follows immediately from E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), Theorem 69.

$$= \frac{1}{x}\left\{\sum_{\alpha_n \leqslant x}' a_n - A(x)\right\} - \left\{\sum_{\alpha_n \leqslant x}' \frac{a_n}{\alpha_n} - \int_1^x A'(t)\frac{dt}{t}\right\} +$$

$$+\left[\left\{\sum_{\alpha_n < T} \frac{a_n}{\alpha_n} - \int_1^T A'(t)\frac{dt}{t}\right\} - \frac{1}{T}\left\{\sum_{\alpha_n < T} a_n - A(T)\right\}\right]. \quad (3.2)$$

Now the first expression in (3.2) is equal to $f(x)$. Hence, making T tend to infinity, we have

$$f(x) - \int_x^\infty f(t)\frac{dt}{t} = \sum_{\alpha_n \leqslant x}' \frac{a_n}{\alpha_n} - \int_1^x A'(t)\frac{dt}{t} - K,$$

where

$$K = \lim_{T\to\infty}\left[\left\{\sum_{\alpha_n < T} \frac{a_n}{\alpha_n} - \int_1^T A'(t)\frac{dt}{t}\right\} - \frac{1}{T}\left\{\sum_{\alpha_n < T} a_n - A(T)\right\}\right].$$

Using a similar notation for b_n, β_n, and $B(x)$, and putting

$$g(x) = \sum_{\beta_n \leqslant x}' \frac{b_n}{\beta_n} - \int_1^x B'(t)\frac{dt}{t} - L,$$

where L is a constant, then, if $g(x)$ belongs to $L^2(0,\infty)$, an argument similar to the above shows that

$$g(x) - \frac{1}{x}\int_0^x g(t)\,dt = \frac{1}{x}\left\{\sum_{\beta_n \leqslant x}' b_n - B(x)\right\}.$$

If we now put $\beta_n = \gamma_n$, $b_n = (2\pi)^{\frac{1}{2}}$, $L = k$, and

$$B(x) = (2\pi)^{\frac{1}{2}}\left\{\frac{x}{2\pi}\log\frac{x}{2\pi} - \frac{x}{2\pi}\right\},$$

then $g(x)$ is equal to $F(x)$ of Theorem 1, and

$$g(x) - \frac{1}{x}\int_0^x g(t)\,dt = \frac{(2\pi)^{\frac{1}{2}}}{x}\left\{N_0(x) - \frac{x}{2\pi}\log\frac{x}{2\pi} + \frac{x}{2\pi}\right\}, \quad (3.3)$$

where $$N_0(x) = \sum_{\gamma_n \leqslant x}' 1 = \tfrac{1}{2}\{N(x-0) + N(x+0)\}.$$

Further, if we put

$$\alpha_n = m\log p, \qquad a_n = \log p/p^{\frac{1}{2}m}, \qquad A(x) = 4\cosh\tfrac{1}{2}x - \phi(x),$$

and $$\phi(x) = \tfrac{1}{2}\log\left(\frac{\tanh\frac{1}{4}x}{\frac{1}{4}x}\right) + \tfrac{1}{4}\pi - \arctan e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}, \quad (3.4)$$

then $f(x)$ is equal to $G(x)$ of Theorem 1, and, by the lemma, the Fourier cosine-transform of (3.3) is accordingly

$$f(x)-\int_x^\infty f(t)\frac{dt}{t}=\sum_{m\log p\leqslant x}{}' \frac{1}{mp^{\frac{1}{2}m}}-2\int_1^x \sinh\tfrac{1}{2}t\frac{dt}{t}-K+\int_1^x \phi'(t)\frac{dt}{t}, \tag{3.5}$$

where

$$K=\lim_{T\to\infty}\Bigg[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}}-2\int_1^T \sinh\tfrac{1}{2}t\frac{dt}{t}-$$

$$-\frac{1}{T}\Bigg\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-4\cosh\tfrac{1}{2}T\Bigg\}+\int_1^T \phi'(t)\frac{dt}{t}-\frac{\phi(T)}{T}\Bigg]$$

$$=\int_1^\infty \phi'(t)\frac{dt}{t}+2\int_0^1 \sinh\tfrac{1}{2}t\frac{dt}{t}+$$

$$+\lim_{T\to\infty}\Bigg[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}}-2\int_0^T \sinh\tfrac{1}{2}t\frac{dt}{t}-\frac{1}{T}\Bigg\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{2}T}\Bigg\}\Bigg].$$

If we write l for the last limit above, then

$$K=\int_1^\infty \phi'(t)\frac{dt}{t}+2\int_0^1 \sinh\tfrac{1}{2}t\frac{dt}{t}+l.$$

Substituting this in (3.5) we have

$$f(x)-\int_x^\infty f(t)\frac{dt}{t}=\sum_{m\log p\leqslant x}{}' \frac{1}{mp^{\frac{1}{2}m}}-2\int_0^x \sinh\tfrac{1}{2}t\frac{dt}{t}-l-\int_x^\infty \phi'(t)\frac{dt}{t}. \tag{3.6}$$

Now the function

$$\int_x^\infty \phi'(t)\frac{dt}{t} \tag{3.7}$$

does not reduce to any simple expression, and it is more convenient to eliminate it. The Fourier cosine-transform of (3.7) is

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \cos xt\,dt\int_t^\infty \phi'(u)\frac{du}{u}$$

$$=\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\sin xt\int_t^\infty \phi'(u)\frac{du}{u}\right]_{t=0}^{t=\infty}+\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \phi'(t)\sin xt\frac{dt}{t}. \tag{3.8}$$

Differentiating (3.4) we find that

$$\phi'(x) = \frac{1}{4}\left(\operatorname{cosech}\tfrac{1}{2}x - \frac{2}{x}\right) + \tfrac{1}{4}\operatorname{sech}\tfrac{1}{2}x - 2e^{-\frac{1}{2}x}.$$

On substituting this in (3.8) the integrated terms vanish, and (3.8) becomes

$$\frac{1}{4x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int\limits_0^\infty \left\{\operatorname{cosech}\tfrac{1}{2}t - \frac{2}{t}\right\}\sin xt\,\frac{dt}{t} + \frac{1}{4x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int\limits_0^\infty \operatorname{sech}\tfrac{1}{2}t\sin xt\,\frac{dt}{t} -$$

$$-\frac{2}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int\limits_0^\infty e^{-\frac{1}{2}t}\sin xt\,\frac{dt}{t}$$

$$= \frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\tfrac{1}{4}I_1 + \tfrac{1}{4}I_2 - 2I_3), \quad \text{say.} \tag{3.9}$$

Now,† if $R(z) > -1$,

$$\log\Gamma(1+z) = \int\limits_0^\infty \left\{ze^{-t} - \frac{1-e^{-zt}}{e^t-1}\right\}\frac{dt}{t}.$$

Putting $z = -\frac{1}{2} \pm ix$ and taking the difference we get

$$\frac{1}{2i}\log\frac{\Gamma(\frac{1}{2}+ix)}{\Gamma(\frac{1}{2}-ix)} = \operatorname{am}\Gamma(\tfrac{1}{2}+ix)$$

$$= \int\limits_0^\infty \left\{xe^{-t} - \frac{e^{\frac{1}{2}t}}{e^t-1}\sin xt\right\}\frac{dt}{t}$$

$$= -\frac{1}{2}\int\limits_0^\infty \left\{\operatorname{cosech}\tfrac{1}{2}t - \frac{2}{t}\right\}\sin xt\,\frac{dt}{t} + \int\limits_0^\infty \left\{xe^{-t} - \frac{\sin xt}{t}\right\}\frac{dt}{t}$$

$$= -\tfrac{1}{2}I_1 + x\int\limits_0^\infty \left\{\frac{\sin t}{t} - \frac{\sin xt}{xt}\right\}\frac{dt}{t} - x\int\limits_0^\infty \left\{\frac{\sin t}{t} - e^{-t}\right\}\frac{dt}{t}. \tag{3.10}$$

Now the second expression in (3.10) is a Frullani integral, and is equal to $x\log x$. Further, for $R(s) > -1$,

$$\int\limits_0^\infty (t^{s-2}\sin t - t^{s-1}e^{-t})\,dt = -\Gamma(s-1)\cos\tfrac{1}{2}s\pi - \Gamma(s)$$

$$= \frac{\Gamma(s+1)}{1-s}\left\{1 - \frac{1-\cos\frac{1}{2}s\pi}{s}\right\}.$$

† This follows from Binet's first integral formula for $\log\Gamma(z)$. Cf. C. A. Stewart, *Advanced Calculus* (London, 1940), 493.

Letting $s \to 0$, we find that

$$\int_0^\infty \left\{\frac{\sin t}{t} - e^{-t}\right\}\frac{dt}{t} = 1,$$

and hence (3.10) gives

$$I_1 = -2\{\operatorname{am}\Gamma(\tfrac{1}{2}+ix) - x\log x + x\}, \tag{3.11}$$

where $\operatorname{am}\Gamma(\frac{1}{2}+ix)$ is defined by putting $\operatorname{am}\Gamma(\frac{1}{2}) = 0$ and continuing analytically along the line $\frac{1}{2}+ix$.

Now†
$$\int_0^\infty \operatorname{sech}\tfrac{1}{2}t\cos xt\,dt = \pi\operatorname{sech}\pi x.$$

Integrating with respect to x we have

$$I_2 = \int_0^\infty \operatorname{sech}\tfrac{1}{2}t\sin xt\frac{dt}{t} = \pi\int_0^x \operatorname{sech}\pi u\,du = \arctan(\sinh\pi x). \tag{3.12}$$

Further
$$I_3 = \int_0^\infty e^{-\frac{1}{2}t}\sin xt\frac{dt}{t} = \arctan 2x. \tag{3.13}$$

Substituting (3.11), (3.12), (3.13) in (3.9) we find that (3.7) and

$$\frac{1}{(2\pi)^{\frac{1}{2}}x}[-\{\operatorname{am}\Gamma(\tfrac{1}{2}+ix) - x\log x + x\} + \tfrac{1}{2}\arctan(\sinh\pi x) - 4\arctan 2x] \tag{3.14}$$

are a pair of Fourier cosine-transforms of $L^2(0,\infty)$. Adding (3.7) to (3.6) and (3.14) to (3.3) we obtain another pair of Fourier cosine-transforms of $L^2(0,\infty)$. These functions are also of bounded variation in any finite interval, and hence it follows as in Theorem 1 that the Fourier integrals concerned converge in the ordinary sense. The result is:

THEOREM 2. *If the Riemann hypothesis is true, and*‡

$$H(x) = \sum_{m\log p\leqslant x}{}' \frac{1}{mp^{\frac{1}{2}m}} - 2\int_0^x \sinh\tfrac{1}{2}t\frac{dt}{t} - l, \tag{3.15}$$

† E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), 177 (7.1.6).

‡ If necessary we may substitute

$$2\int_0^x \sinh\tfrac{1}{2}t\frac{dt}{t} = \operatorname{li}(e^{\frac{1}{2}x}) - \operatorname{li}(e^{-\frac{1}{2}x})$$

in (3.15).

where

$$l = \lim_{T\to\infty}\left[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}} - 2\int_0^T \sinh\tfrac{1}{2}t\frac{dt}{t} - \frac{1}{T}\left\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}} - 2e^{\frac{1}{2}T}\right\}\right], \tag{3.16}$$

and

$$K(x) = \frac{(2\pi)^{\frac{1}{2}}}{x}\left\{N_0(x) - \frac{1}{2\pi}\operatorname{am}\Gamma(\tfrac{1}{2}+ix) + \frac{x}{2\pi}\log 2\pi + \right.$$
$$\left. + \frac{1}{4\pi}\arctan(\sinh\pi x) - \frac{2}{\pi}\arctan 2x\right\},$$

then, for $x > 0$, $$H(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty} K(t)\cos xt\,dt,$$

and $$K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty} H(t)\cos xt\,dt. \tag{3.17}$$

4. The formula for $N(x)$

As before, putting $\alpha_n = m\log p$, $\alpha_N = T$, $a_n = \log p/p^{\frac{1}{2}m}$, the right-hand side of (3.17) is

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty}\left\{\sum_{\alpha_n\leqslant t}{}' \frac{a_n}{\alpha_n} - 2\int_0^t \sinh\tfrac{1}{2}u\frac{du}{u} - l\right\}\cos xt\,dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{N\to\infty}\left[\sum_{n=1}^{N-1}\left(\frac{a_1}{\alpha_1}+\frac{a_2}{\alpha_2}+\dots+\frac{a_n}{\alpha_n}\right)\int_{\alpha_n}^{\alpha_{n+1}}\cos xt\,dt - \right.$$
$$\left. -2\int_0^T \cos xt\,dt\int_0^t \sinh\tfrac{1}{2}u\frac{du}{u} - \frac{l}{x}\sin xT\right]$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{N\to\infty}\left[\frac{1}{x}\sum_{n=1}^{N-1}\left(\frac{a_1}{\alpha_1}+\frac{a_2}{\alpha_2}+\dots+\frac{a_n}{\alpha_n}\right)(\sin x\alpha_{n+1} - \sin x\alpha_n) - \right.$$
$$\left. -\frac{2}{x}\int_0^T \sinh\tfrac{1}{2}u(\sin xT - \sin xu)\frac{du}{u} - \frac{l}{x}\sin xT\right]$$

$$= \frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{N\to\infty}\left[-\sum_{n=1}^{N}\frac{a_n}{\alpha_n}\sin x\alpha_n + \right.$$
$$\left. + \sin xT\left\{\sum_{n=1}^{N}\frac{a_n}{\alpha_n} - 2\int_0^T \sinh\tfrac{1}{2}u\frac{du}{u} - l\right\} + 2\int_0^T \sinh\tfrac{1}{2}u\sin xu\frac{du}{u}\right].$$

By (3.16) this is equal to

$$\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{N\to\infty}\left[-\sum_{n=1}^{N}\frac{a_n}{\alpha_n}\sin x\alpha_n+\int_0^T e^{\frac{1}{2}u}\sin xu\frac{du}{u}+\right.$$
$$\left.+\frac{\sin xT}{T}\left\{\sum_{n=1}^{N}a_n-2e^{\frac{1}{2}T}\right\}\right]-\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\frac{1}{2}u}\sin xu\frac{du}{u}$$
$$=-\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{T\to\infty}\left[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}}\sin(xm\log p)-\int_0^T e^{\frac{1}{2}u}\sin xu\frac{du}{u}-\right.$$
$$\left.-\frac{\sin xT}{T}\left\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{2}T}\right\}\right]-\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\arctan 2x,$$

and by (3.17) this expression is equal to $K(x)$. Substituting the value of $K(x)$ from Theorem 2 and rearranging the terms, we obtain the result:

THEOREM 3.† *If the Riemann hypothesis is true and $x \geqslant 0$, then*

$$\tfrac{1}{2}\{N(x-0)+N(x+0)\}-\left(\frac{x}{2\pi}\log\frac{x}{2\pi}-\frac{x}{2\pi}\right)$$
$$=-\frac{1}{\pi}\lim_{T\to\infty}\left[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}}\sin(xm\log p)-\int_0^T e^{\frac{1}{2}u}\sin xu\frac{du}{u}-\right.$$
$$\left.-\frac{\sin xT}{T}\left\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{2}T}\right\}\right]+\frac{1}{2\pi}\{\operatorname{am}\Gamma(\tfrac{1}{2}+ix)-x\log x+x\}-$$
$$-\frac{1}{4\pi}\arctan(\sinh \pi x)+\frac{1}{\pi}\arctan 2x,$$

where $\operatorname{am}\Gamma(\frac{1}{2}+ix)$ *is defined by taking* $\operatorname{am}\Gamma(\frac{1}{2}) = 0$ *and continuing* $\Gamma(s)$ *along the line* $s = \frac{1}{2}+ix$.

Theorem 3 is, in a sense, analogous to (2.5), and the argument of this section can be reversed to deduce Theorem 2 from Theorem 3.

5. Simpler pairs of transforms

The pairs of transforms discussed in §§ 2, 3 have simpler analogues with regularly spaced discontinuities. For example, the functions

$$\frac{1}{x}\left\{\sum_{n\leqslant x}{}' 1-x\right\},\qquad C-\left\{\sum_{n\leqslant x}{}'\frac{1}{n}-\log x\right\} \tag{5.1}$$

† See (A), Theorem 2, for an alternative proof and discussion of the result.

are a pair of transforms with respect to the kernel $2\cos 2\pi x$. Further, if $\chi(n)$ is a real primitive character *modulo* κ ($\kappa > 1$) and

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n} = S,$$

then the functions

$$\frac{1}{x}\sum_{n\leqslant x}{}' \chi(n), \qquad S - \sum_{n\leqslant x}{}' \frac{\chi(n)}{n}$$

are a pair of transforms with respect to the kernel $2\kappa^{-\frac{1}{2}}\cos(2\pi x/\kappa)$ if $\chi(-1) = 1$, or with respect to the kernel $2\kappa^{-\frac{1}{2}}\sin(2\pi x/\kappa)$ if

$$\chi(-1) = -1.$$

These results are easily proved by the method of § 2, using an ordinary Fourier series in the place of (2.5). The pair of transforms (5.1) can also be derived as a limiting case of an earlier result.†

† A. P. Guinand, *J. of London Math. Soc.* 14 (1939), 97–100. Let $\beta \to 1-0$ in (1).